Unique continuation at the boundary for harmonic functions

Xavier Tolsa

PDE Seminar, July 9, 2020

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Yes, if $n \geq 2$ and both $u = \nabla u = 0$ in an *open* subset of $\partial\Omega$.
Proved using that $\Delta u = (c\, \partial_\nu u)\sigma$ in that open subset of $\partial\Omega$.

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No, for $n \geq 3$, in general.
Bourgain and Wolff (1990) showed that there exists a function $u : \overline{\mathbb{R}^n_+} \to \mathbb{R}$ harmonic in $\mathbb{R}^n_+$ and C^1 in $\overline{\mathbb{R}^n_+}$, such that $u = \nabla u = 0$ in a subset of $\partial\mathbb{R}^n_+$ of positive measure and $u \not\equiv 0$ in $\overline{\mathbb{R}^n_+}$.

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Still open the case of C^2 or C^∞ functions.

The question of Fang-Hua Lin (1991)

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain.
Let $u : \overline{\Omega} \to \mathbb{R}$ be harmonic in Ω and continuous in $\overline{\Omega}$.
Suppose that $u = 0$ in a relatively open subset $\Sigma \subset \partial\Omega$ and that $\nabla u = 0$ in a subset of Σ of positive surface measure.
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- Fang-Hua Lin and Adolfsson - Escauriaza: If $u = 0$ in Σ and $u \not\equiv 0$ in Ω, then ∇u vanishes in Σ at most in a subset of dimension $n - 2$.

The main result

Theorem (T., 2020)

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Let B be a ball centered in $\partial\Omega$, and suppose that $\Sigma = B \cap \partial\Omega$ is a Lipschitz graph with small enough constant (depending on n).
Let $u : \overline{\Omega} \to \mathbb{R}$ be harmonic in Ω and continuous in $\overline{\Omega}$. Suppose that $u = 0$ in Σ and that $\nabla u = 0$ in a subset of Σ of positive surface measure.
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Remarks:

- So the result is also true for C^1 domains. Open up to now.
- No results about the dimension of the set where ∇u may vanish when $u \not\equiv 0$.

A corollary about harmonic measure

Corollary

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, let B be a ball centered in $\partial\Omega$, and suppose that $\Sigma = B \cap \partial\Omega$ is a Lipschitz graph with small enough constant. Let ω^p, ω^q be the harmonic measures for Ω with poles in p, q.

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Suppose that there exists some subset $E \subset \Sigma$ *with positive harmonic measure such that*

$$\omega^p|_E = \omega^q|_E$$

Then $p = q$.

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$$\omega^p|_E = \omega^q|_E$$

Then $p = q$.

Remarks:

- Saying that $\omega^p|_E = \omega^q|_E$ is the same as saying that

$$\omega^p(F) = \omega^q(F) \quad \text{for all } F \subset E.$$

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Remarks:

- The corollary follows by applying the theorem to

$$u = g(\cdot, p) - g(\cdot, q) \quad \text{in } \Omega \setminus (\overline{B}(p, \varepsilon) \cup \overline{B}(q, \varepsilon)),$$

  taking into account that $\omega \approx \sigma$ and that $\frac{d\omega^p}{d\sigma} = -\partial_\nu g(\cdot, p)$.

Ideas for the proof: doubling properties of u

Most results about unique continuation of harmonic functions (and solutions of other elliptic PDE's) follow by studying their doubling properties.

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Most results about unique continuation of harmonic functions (and solutions of other elliptic PDE's) follow by studying their doubling properties.
Denote

$$H(x,r) = \int_{\partial B(x,r)} |u|^2 \, d\sigma.$$

Theorem (Adolfsson, Escauriaza, Kenig)

Let Ω be a Lipschitz domain, Σ open in $\partial\Omega$, and u harmonic in Ω, continuous in $\overline{\Omega}$, such that $u = 0$ in Σ and $u \not\equiv 0$ in Ω. Suppose that

$$\frac{H(x,2r)}{H(x,r)} \leq C \qquad \text{for all } x \in \Sigma,\ 0 < r \leq r_0.$$

Then $|\partial_\nu u|$ is (locally) an A_∞ weight, and thus $\partial_\nu u$ cannot vanish in a subset of positive measure in Σ.

Ideas for the proof: doubling properties of u (2)

Recall

$$H(x,r) = \int_{\partial B(x,r)} |u|^2 \, d\sigma.$$

A pointwise result:

Theorem (Adolfsson, Escauriaza)

Let Ω be a Lipschitz domain, Σ open in $\partial\Omega$, and u harmonic in Ω, continuous in $\overline{\Omega}$, such that $u = 0$ in Σ and $u \not\equiv 0$ in Ω.
Suppose that $x \in \Sigma$ is a density point of $\{x : \partial_\nu u = 0\}$. Then,

$$\lim_{r \to 0} \frac{H(x,2r)}{H(x,r)} = \infty,$$

and u vanishes to ∞ order in x.

Carleman inequalities and the frequency function

There are two typical approaches to study the doubling properties of u:

- By Carleman inequalities.
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Carleman inequalities seem more flexible from the PDE point of view.

The use of the Almgren frequency function seems more appropriate for results involving geometric arguments.

Recently used by Cheeger, Naber and Valtorta to obtain effective estimates for the size of the singular and critical sets for solution of elliptic PDE's, and by Logunov in his work on the Nadirashvili conjecture.

The frequency function

We will use the frequency function:

$$N(x,r) = r\,\partial_r \log \fint_{\partial B(x,r)} |u|^2\,d\sigma = r\,\partial_r \log \frac{H(x,r)}{\sigma(\partial B(x,r))}.$$

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By integration by parts (assuming $B(x,r) \cap \partial\Omega \subset \Sigma$):

$$N(x,r) = \frac{2r \int_{B(x,r)} |\nabla u|^2\,dy}{\int_{\partial B(x,r)} |u|^2\,d\sigma} = \frac{2r \int_{B(x,r)} |\nabla u|^2\,dy}{H(x,r)}.$$

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If u is a harmonic d-homogeneous polynomial, then $N(0,r) = 2d$ for all r.

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$$N(x,r) \leq N(x,r_0) < \infty \quad \text{if } 0 < r < r_0 \text{ and } B(x,r_0) \subset \Omega.$$

Then,

$$\partial_r \log \fint_{\partial B(x,r)} |u|^2 \, d\sigma = \frac{N(x,r)}{r} \leq \frac{N(x,r_0)}{r}.$$

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Integrating between $r/2$ and r,

$$\log \frac{\fint_{\partial B(x,r)} |u|^2 \, d\sigma}{\fint_{\partial B(x,r/2)} |u|^2 \, d\sigma} \leq N(x,r_0) \log 2.$$

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So,

$$\limsup_{r \to 0} \frac{H(x,2r)}{H(x,r)} < \infty.$$

The frequency function (3)

When $B(x,r) \cap \partial\Omega \neq \varnothing$, the situation is more complicated. We have:

$$\partial_r N(x,r) = (\cdots) + \frac{2}{H(x,r)} \int_{B(x,r)\cap\partial\Omega} (y-x)\cdot\nu(y) \left|\partial_\nu u(y)\right|^2 d\sigma(y),$$

where $(\cdots) \geq 0$.

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This is equivalent to say that $B(x,r)\cap\overline{\Omega}$ is star-shaped with respect to x.

The condition $(y - x) \cdot \nu(y) \geq 0$

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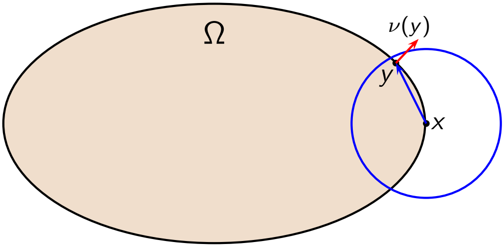
$$(y - x) \cdot \nu(y) \geq 0 \quad \text{for all } y \in B(x, r) \cap \partial\Omega$$

holds for convex domains and $x \in \partial\Omega$:



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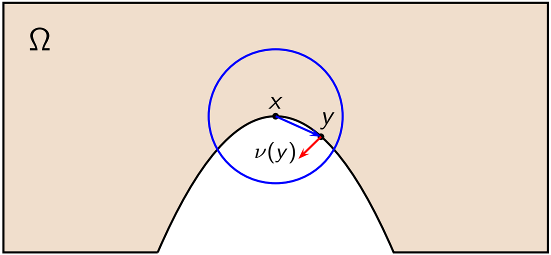
may fail for Lipschitz domains and $x \in \partial\Omega$:

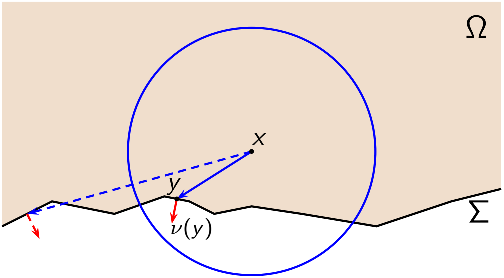


The condition $(y-x)\cdot\nu(y)\geq 0$ at points x close to $\Sigma\subset\partial\Omega$

$\Omega\subset\mathbb{R}^n$ Lipschitz domain.
If $x\in\Omega$, $B(x,r)\cap\partial\Omega\subset\Sigma$, the slope of Σ is $\leq\theta$, and $\operatorname{dist}(x,\Sigma)\geq C\theta\, r$, then

$$(y-x)\cdot\nu(y)\geq 0 \quad \text{for all } y\in B(x,r)\cap\partial\Omega$$

and thus $\partial_r N(x,r)\geq 0$.



Approaching points in $\partial\Omega$ from the interior

We would like to show that

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Given $x \in \Sigma$, if there exists a sequence $\{x_k\}_k \subset \Omega$ such that

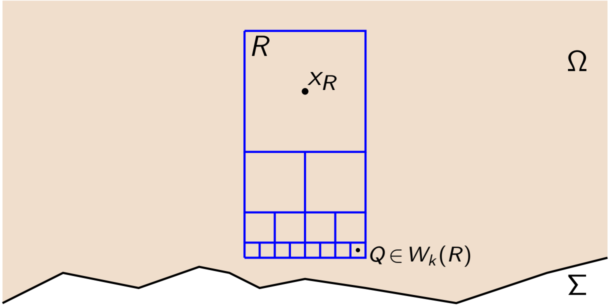
$$|x_k - x| \approx \operatorname{dist}(x_k, \Sigma) \to 0$$

and $N(x_k, 100|x_k - x|) \leq C$ for all k, then, for $r_k = |x - x_k|$,

$$\frac{H(x,2r_k)}{H(x,r_k)} \lesssim \frac{H(x_k,4r_k)}{H(x_k,r_k/2)} \leq 8^{N(x_k,100|x_k-x|)} \leq C,$$

and thus (1) holds.

Some Whitney cubes



The Key Lemma

Lemma

Let $R \subset \Omega$ be a Whitney cube. Let $W_k(R)$ be the family of Whitney cubes at k-levels down from R, as above. Suppose that $N(x_R, 100\ell(R)) \geq N_0$. Let $\delta \in (0, 10^{-3})$. If k is big enough, then at least 10% of the cubes $Q \in W_k(R)$ satisfy

$$N(x_Q, \delta^{-2}\ell(Q)) \leq \frac{1}{2}\, N(x_R, \delta^{-2}\ell(R)).$$

The remaining 90% cubes from $W_k(R)$ satisfy

$$N(x_Q, \delta^{-2}\ell(Q)) \leq (1+\delta)\, N(x_R, \delta^{-2}\ell(R)).$$

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Lemma

Let $R \subset \Omega$ be a Whitney cube. Let $W_k(R)$ be the family of Whitney cubes at k-levels down from R, as above. Suppose that $N(x_R, 100\ell(R)) \geq N_0$. Let $\delta \in (0, 10^{-3})$. If k is big enough, then at least 10% of the cubes $Q \in W_k(R)$ satisfy

$$N(x_Q, \delta^{-2}\ell(Q)) \leq \frac{1}{2}\, N(x_R, \delta^{-2}\ell(R)).$$

The remaining 90% cubes from $W_k(R)$ satisfy

$$N(x_Q, \delta^{-2}\ell(Q)) \leq (1+\delta)\, N(x_R, \delta^{-2}\ell(R)).$$

- The proof uses techniques developed by Logunov in connection with Nadirashvili's conjecture.
- Monotonicity of $N(x, \cdot)$ for x close Σ and combinatorial arguments.
- It relies on a result of quantitative unique Cauchy continuation.

Application of the law of large numbers

For $x \in \Sigma$, consider a sequence of Whitney cubes $\{Q_j\}$ centered at $\{x_j\}$ approaching x with $\ell(Q_j) \approx \text{dist}(x, Q_j)$, $\ell(Q_j) = 2^{-kj}$. By the law of large numbers, for σ-a.e. $x \in \Sigma$, about 10% of the cubes in the sequence satisfy

$$N(x_j, \delta^{-2}\ell(Q_j)) \leq \frac{1}{2}\, N(x_{j-1}, \delta^{-2}\ell(Q_{j-1})),$$

and the other 90% of cubes satisfy

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Thus

$$\liminf_{r\to 0} \frac{H(x,2r)}{H(x,r)} < \infty \quad \text{for } \sigma\text{-a.e. } x \in \Sigma,$$

and by Adolfsson-Escauriaza x is not a density point of $\{x \in \Sigma : \nabla u = 0\}$.

Open problems

- Is the answer to Fang-Hua Lin's problem positive for general Lipschitz domains?

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- And for chord-arc domains (i.e., NTA domains with AD-regular boundary)?

- Can one get any information of the blowups of u and Ω at the boundary?
  The case of convex domains studied by McCurdy (2019).

Thank you!