Batchelor's law for passive scalar turbulence

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Turbulence

Observed in a variety of weakly dissipative physical systems (e.g., fluids, plasmas). Key features:

- **Chaotic**: Extreme sensitivity to initial data, positive Lyapunov exponent.
- **Ergodic**: Time averages the same as running multiple experiments.
- **Multiscale**: Exhibits a multitude of scales and a cascade between scales.
- **Universality**: Statistics between the largest and smallest scales appear to be *universal*, i.e., independent of details of experiment.

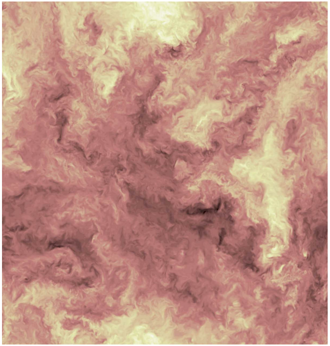


Figure: Credit: Johns Hopkins turbulence data base

Turbulence

Example: hydrodynamic turbulence in the Navier-Stokes equations: describe dynamics of

$$\partial_t u + (u \cdot \nabla) u - \nabla p = \nu \Delta u + F$$

as $\nu \to 0$.

- Inherently high/infinite dimensional: number of 'active' modes $\to \infty$ as $\nu \to 0$
- Beyond well-posedness challenges: quantitative dynamical information on how energies are transferred from large spatial scales (low modes) to small spatial scales
- Major open problems, not at all settled (even among physicists– intermittency corrections to K41 predictions)

Topic of this talk: more tractable case of **passive scalar turbulence**.

Passive scalar advection

Setting: Incompressible fluid on a domain $\Omega \subset \mathbb{R}^d$ or $\Omega = \mathbb{T}^d, d = 2, 3$ with velocity field $u(t,x), x \in \Omega, t \geq 0$ (e.g., solution to Navier-Stokes with **fixed** viscosity $\nu > 0$)

Passive scalar advection with source[1] G and diffusivity $\kappa > 0$:

$$\partial_t g + \underbrace{u \cdot \nabla g}_{\text{Advection by } u(t,x)} = \underbrace{\kappa \Delta g}_{\text{Diffusivity}} + \underbrace{G}_{\text{Source}}, \quad g(0,x) = g_0(x)$$

[1] $\int G(t,x)dx \equiv 0$

Batchelor-regime passive scalar turbulence

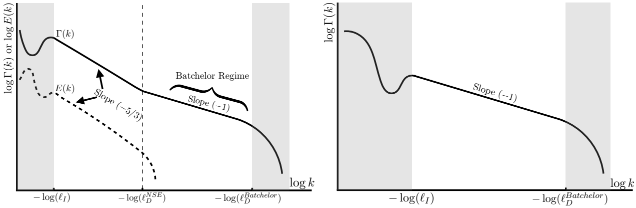
$$u(t,\cdot):\Omega\to\mathbb{R}^d, \quad \nabla\cdot u\equiv 0$$

$$\partial_t g + u\cdot\nabla g = \kappa\Delta g + G\,, \quad g(0,x)=g_0(x)$$

- Physicists often study the *power spectral density* of the ensemble average (defined precisely later!)

$$\Gamma(k) := \mathbf{E}|k|^{d-1}|\hat{g}(k)|^2 \quad \text{and} \quad E(k) := \mathbf{E}|k|^{d-1}|\hat{u}(k)|^2$$

- No mathematically rigorous proof of any power spectrum in fluid mechanics.
- In 1959, Batchelor predicted a spectrum of $\Gamma(k)\approx|k|^{-1}$ in the regime $\nu\gg\kappa$. $\mathrm{Sc}=\nu/\kappa$ is called the Schmidt number.



Observation, experiments, and numerics

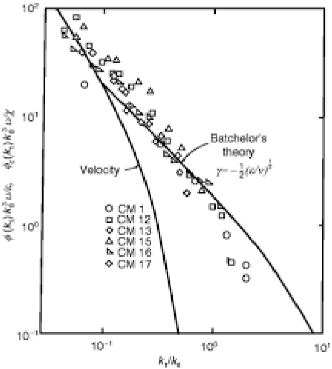


Figure: Gibson, C., and W. H. Schwarz. "The universal equilibrium spectra of turbulent velocity and scalar fields." Journal of Fluid Mechanics 16.3 (1963): 365-384. Spectra for salinity concentrations in grid-driven turbulence experiment.

Observation, experiments, and numerics

Comparison of modern numerical experiments.

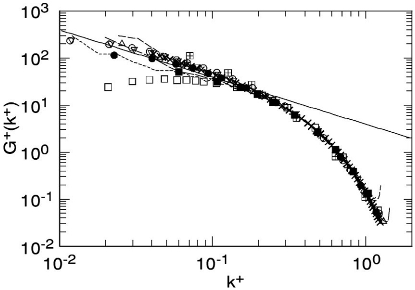


Figure: Antonia, R. A., and P. Orlandi. "Effect of Schmidt number on small-scale passive scalar turbulence." Appl. Mech. Rev. 56.6 (2003): 615-632. Comparison of spectra generated by various numerical calculations by a variety of authors.

Small scales in passive scalar advection

Incompressible fluid on a domain $\Omega \subset \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$, $d = 2, 3$ with velocity field $u(t, x), x \in \Omega, t \geq 0$.

Passive scalar advection with source[2] G and diffusivity $\kappa > 0$:

$$\partial_t g + u \cdot \nabla g = \kappa \Delta g + G\,, \quad g(0, x) = g_0(x)$$

Key point, known to physicists: Creation of small scales in g due to *chaotic properties of Lagrangian flow* ϕ^t on Ω,

$$\frac{d}{dt}\phi^t = u(t, \phi^t(x))$$

[2] $\int G(t,x)dx \equiv 0$

Results in this paper: proof of Batchelor power law

$$u(t,\cdot):\Omega\to\mathbb{R}^d, \quad \nabla\cdot u\equiv 0, \quad \frac{d}{dt}\phi^t(x)=u(t,\phi^t(x))$$
$$\partial_t g + u\cdot\nabla g = \kappa\Delta g + G\,, \quad g(0,x)=g_0(x)$$

Our series of four papers: when $u(t,x)$ evolves by stochastic Navier-Stokes on $\Omega=\mathbb{T}^d$, $d=2$ or $d=3$: [3]

- Lagrangian flow ϕ^t is chaotic (sensitivity w.r.t. initial conditions, exponentially fast mixing)
- Rigorous proof of Batchelor's 1959 law for power spectrum along inertial range for statistically stationary passive scalars as $\kappa\to 0$

[3] For $d=3$ our results apply when u evolves by a hyperviscous regularization of stochastic 3D NSE.

Mechanism for generating small scales

$$u(t,\cdot) : \Omega \to \mathbb{R}^d, \quad \nabla \cdot u \equiv 0, \quad \frac{d}{dt}\phi^t(x) = u(t, \phi^t(x))$$
$$\partial_t g + u \cdot \nabla g = \kappa \Delta g + G\,, \quad g(0,x) = g_0(x)$$

At $\kappa = 0$, $G \equiv 0$, have $g(t,x) = g_0((\phi^t)^{-1}(x))$. Using $\nabla \cdot u \equiv 0$:

$$\|\nabla g(t,\cdot)\|_{L^2}^2 = \int |(D_x\phi^t)^{-\top}(\nabla_x g_0)|^2 \, dx\,.$$

Growth of $|(D_x\phi^t)^{-\top}| = |D_x\phi^t| \Rightarrow \|g(t,\cdot)\|_{H^1} \to \infty$

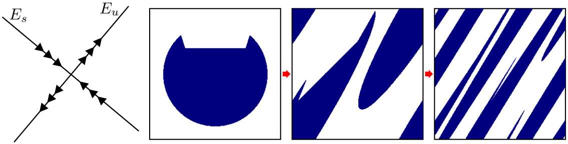
Definition (Lyapunov exponent)

$$\lambda(u,x) = \limsup_{t\to\infty} \frac{1}{t} \log |D_x\phi^t|$$

Positive Lyapunov exponent implies **chaotic dynamics** exhibiting **sensitivity with respect to initial conditions**. Necessary but not sufficient for **fast mixing**.

(Dynamical) Hyperbolicity: local mechanism for chaos

CAT map $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x \pmod{\mathbb{Z}^2}$.



- Sensitivity w.r.t. initial conditions:
  $d(F^n(p_1), F^n(p_2)) \gtrsim e^{\alpha n} d(p_1, p_2)$ when $p_1 - p_2 \notin E^s$
- Fast mixing: for scalars $\phi, \psi \in H^1$,

$$\left| \int \phi \cdot \psi \circ F^n - \int \phi \int \psi \right| \le C \|\phi\|_{H^1} \|\psi\|_{H^1} e^{-\beta n},$$

$\alpha, \beta, C > 0$ constants.

Well-known: these hold for all *uniformly hyperbolic* systems

Heuristic for −1 power law

Illustration of role played by **hyperbolicity**:

- Consider CAT map $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F(x) = Ax \pmod{\mathbb{Z}^2}$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
- Discrete-time toy model of passive scalar "advection": (ignores diffusivity for now)

$$g_{n+1}(x) = g_n \circ F^{-1}(x) + \omega_{n+1} \sin(2\pi x_1)\,,$$

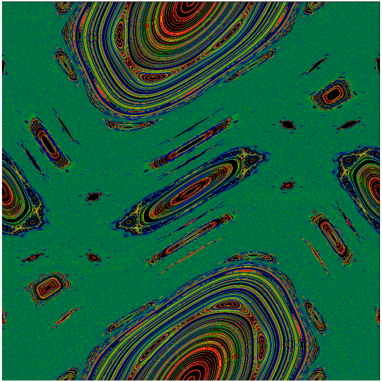
  $\{\omega_j\}$ IID.
- With $g_0 \equiv 0$, have

$$g_n(x) = \sum_{j=0}^{n-1} \omega_{n-j} \sin 2\pi \left\langle x, (A^\top)^{-j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle .$$

- Let $\lambda > 1 > \lambda^{-1}$ be eigenvalues of A. Then, $\mathbb{E}\|\Pi_{\lambda^m \le \cdot \le \lambda^{m+1}} g_n\|^2 \approx 1$ for all m, consistent with −1 power law.

Beyond uniform hyperbolicity

Typically, hyperbolicity *not uniform*. Most systems of physical interest have "mixed" behavior: **elliptic** and **hyperbolic**

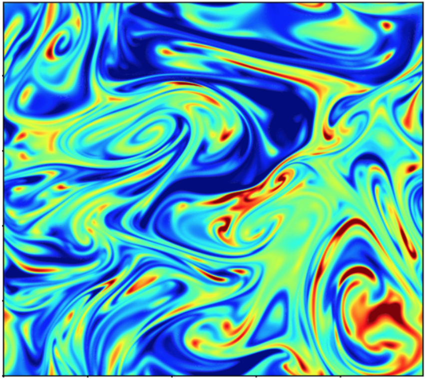


Picture credit: Wikipedia user Linas

- At left: Chirikov standard map $F : \mathbb{T}^2 \to \mathbb{T}^2$
- **Hyperbolicity** at $p \in \mathbb{T}^2$ in green region, where Lyapunov exponent $\lambda(p) = \limsup_{n\to\infty} \frac{1}{n} \log |D_p F^n|$ is positive.
- Standard map conjecture: $\{\lambda(p) > 0\}$ has positive area. **Wide open.**

Deterministic Lagrangian flow

$$u(t,\cdot) : \mathbb{T}^d \to \mathbb{R}^d\,, \quad \nabla \cdot u \equiv 0\,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$



Chirikov standard map a **toy model** for stretching and folding generating small scales.

Hopelessly out of reach to prove Lagrangian chaos for deterministic fluids models.

Picture credit: Paul Götzfried, Mohammad S. Emran, Emmanuel Villermaux, and Jörg Schumacher, Phys. Rev. Fluids 4, 2019

Stochastic Navier-Stokes model

$$u(t,\cdot):\mathbb{T}^d\to\mathbb{R}^d\,,\quad \nabla\cdot u\equiv 0\,,\quad \dot\phi^t(x)=u(t,\phi^t(x))$$

- Problem is **tractable** in the presence of noise!
- Consider, e.g., 2D Navier-Stokes with stochastic forcing:

$$\partial_t u+(u\cdot\nabla)u+\nabla p=\nu\Delta u+Q\dot W_t\,,\quad \nabla\cdot u\equiv 0$$

where QW_t is white-in-time, mean zero, divergence free, spatially Sobolev

  - 2D Navier-Stokes globally (mildly) well-posed for a.e. path realization
  - Markov process $u_t=u(t,\cdot)$; unique stationary measure when QW_t "sufficiently nondegenerate" (e.g., Flandoli-Maslowski, Hairer-Mattingly, Kuksin-Shirikyan)

1. Lagrangian Chaos

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q\dot{W}_t\,, \quad \nabla \cdot u \equiv 0\,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

Theorem (BBPS 2018, submitted)

If QW_t satisfies certain nondegeneracy condition, then $\exists$ **deterministic constant** $\lambda > 0$ *such that*

$$\lim_{t\to\infty} \frac{1}{t} \log |D_x \phi^t| = \lambda > 0 \qquad \textit{w.p.1}$$

for all initial $x \in \mathbb{T}^2$ and Sobolev regular vector fields u_0.

Same for 3D hyperviscous NSE, 2D & 3D Stokes and Galerkin-NSE.

Proof a combination of random dynamical systems theory (Furstenberg rigidity principle) with SPDE analysis

1. Lagrangian Chaos

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Nondegeneracy needed is very mild: result valid for u_t given by

$$u_t(x,y) = \begin{pmatrix} Z_1(t)\sin y + Z_2(t)\cos y \\ Z_3(t)\sin x + Z_4(t)\cos x \end{pmatrix},$$

$dZ_i = -Z_i dt + dW_t^{(i)}$ independent Ornstein-Uhlenbeck processes.

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Same for 3D hyperviscous NSE, 2D & 3D Stokes and Galerkin-NSE.

Corollary: for solutions to $\partial_t g + u \cdot \nabla g = 0$, have $\|g(t,\cdot)\|_{H^1} \gtrsim e^{\lambda t}$.
Generation of small scales in passive scalar!

2. Almost-sure exponential mixing

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

Theorem (BBPS 2019, submitted)

Under the same conditions as previous theorem, for all $p \geq 1$, there exists a **deterministic** $\gamma = \gamma(p) > 0$ *and a random constant* $C = C(\omega, u_0, p)$ *such that* $\mathbb{P} \times \mu$ *a.e.* (ω, u_0) *and arbitrary mean-zero* $f \in H^1(\mathbb{T}^d)$*, we have*

$$\left| \int f(x) \cdot g \circ \phi^t(x) \, dx \right| \leq C e^{-\gamma t} \| f \|_{H^1} \| g \|_{H^1}$$

with $\mathbb{E} \int C^p d\mu(u_0) < \infty$.

- Corollary: exponential H^{-1} decay for solutions to $\partial_t g + u \cdot \nabla g = 0$.
- A priori much stronger than positive Lyapunov exponent. Proof uses previous theorem as a lemma.

3. L^2 enhanced dissipation

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + Q \dot{W}_t \,, \quad \nabla \cdot u \equiv 0 \,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))$$

Cascade of L^2 mass to higher modes- should **strengthen** effect of diffusivity $\kappa\Delta$ for solutions to $\partial_t g + u \cdot \nabla g = \kappa \Delta g$.

Theorem (BBPS19-II, submitted)

For all L^2 initial $g(0,x) = g_0(x)$ with $\int g_0 = 0$, have

$$\|g(t,\cdot)\|_{L^2} \lesssim e^{-c|\log\kappa|^{-1} t} \|g_0\|_{L^2} \quad w.p.1$$

- $|\log \kappa|$ timescale for dissipation is **sharp** for C^2-regular velocity fields.
- NOT a corollary of previous work: requires correlation decay for *stochastic representations*

$$\frac{d}{dt}\phi^t_\kappa = u(t, \phi^t_\kappa(x)) + \sqrt{\kappa}\widehat{\dot{W}_t}$$

- C.f. stochastic stability of Ruelle resonances: Blank-Keller-Liverani '02 (Anosov maps), Dyatlov-Zworski '16 (contact Anosov flows)

4. Batchelor's law for passive scalar turbulence

Batchelor regime: fluid evolution $u(t, \cdot)$ fixed, $\kappa \to 0$ in passive scalar advection

$$\partial_t g^\kappa + u \cdot \nabla g^\kappa = \kappa \Delta g^\kappa + \eta \dot{\widetilde{W}}_t$$

Theorem (BBPS19-III, submitted)

Let (u, g^κ) be statistically stationary. Then,

$$\mathbf{E} \| \Pi_{\leq N} g^\kappa \|_{L^2}^2 \approx \log N \quad \textit{for} \quad 1 \ll N \lesssim \kappa^{-1/2}$$

where $\Pi_{\leq N} g$ is projection onto Fourier modes $\sin(k \cdot x), \cos(k \cdot x)$, $|k|_\infty \leq N$

Consistent with power law $\Gamma(k) \approx |k|^{-1}$, $\Gamma(k) := |k|^{d-1} \mathbf{E} |\hat{g}(k)|^2$

Ideas from the proof

Diverse array of tools needed:

- Dynamics:
  - Multiplicative ergodic theory
  - Random dynamical systems
- Stochastics:
  - Malliavin calculus / nonadapted stochastic calculus for infinite-dimensional systems
  - Lyapunov/drift conditions: correlation decay for Markov chains on "large" systems

Toy model: IID compositions of matrices

- Consider compositions $A_n \cdots A_2 A_1$ of IID determinant 1 matrices $A_i, i \geq 1$

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  - Random shear: $A_1 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ with e.g. $s \sim N(0,1)$;
  - Stretching and compression get twisted back in on each other:

$$A_1 = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \text{with probability } p \in (0,1) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{with probability } 1-p \in (0,1). \end{cases}$$

Furstenberg '68: these are essentially the only cases. Staggeringly strong rigidity result using the algebraic structure of $SL_2(\mathbb{R})$.

Theorem (Furstenberg '68)

If $\eta = 0$ then one of two cases:

(a) $\exists$ *inner product* $\langle \cdot, \cdot \rangle$ *with respect to which* A_1 *is almost-surely an isometry.*

(b) $\exists$ *lines* $\{L_i\}_{i=1}^{p}, p \in \{1, 2\}$ *such that for all* $1 \leq i \leq p$, *have* $A_1 L_i = L_j$ *for some* j.

General principle of Furstenberg criteria: if $\lambda = 0$ then matrices A_i has an "almost-surely invariant structure".

Back to Lagrangian flow: ∃ Lyapunov exponent λ

$$u(t,\cdot) : \mathbb{T}^d \to \mathbb{R}^d , \quad \dot{\phi}^t(x) = u(t, \phi^t(x)) , \quad u_t = u(t,x) , \quad x_t = \phi^t(x_0) .$$

Lemma (Application of Oseledets' Multiplicative Ergodic Theorem)

Assume (u_t, x_t) has a unique stationary measure. Then, $\exists \lambda \geq 0$ deterministic constant such that

$$\lim_{t\to\infty} \frac{1}{t} \log |D_x \phi^t| = \lambda \quad w.p.1$$

for all Sobolev regular $u_0 = u(0,\cdot)$ and $x_0 \in \mathbb{T}^d$.

- Large lit. on ergodicity / uniqueness of stat. measures for (u_t) process. In our setting stat. measure μ for (u_t) unique by Flandoli-Maslowski if noise nondegenerate, Sobolev-regular.
- Process (u_t, x_t) requires some more work- always hypoelliptic, even when (u_t) noise is completely nondegenerate.

Proof of $\lambda > 0$ by contradiction

$$u(t,\cdot) : \mathbb{T}^d \to \mathbb{R}^d\,, \quad \dot{\phi}^t(x) = u(t, \phi^t(x))\,, \quad u_t = u(t,x)\,, \quad x_t = \phi^t(x_0)\,.$$

Proposition (BBPS 18)

Fix $d = 2$. If $\lambda = 0$, 2 cases:

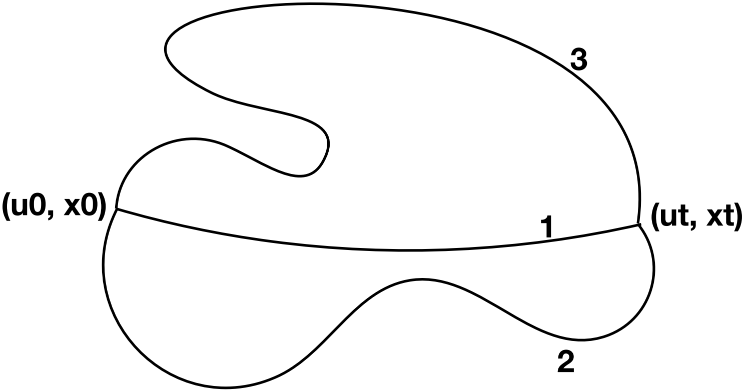
(a) $\exists$ *deterministic, continuously-varying family of inner products* $\langle \cdot, \cdot \rangle_{u,x}$ *such that* $D_{x_0}\phi^t$ *an isometry* $\langle \cdot, \cdot \rangle_{u_0,x_0} \to \langle \cdot, \cdot \rangle_{u_t,x_t}$.

(b) $\exists$ *deterministic, continuously-varying families of lines* $L^i(u,x), i \le p, p = 1, 2$ *such that*

$$D_{x_0}\phi^t\left(\cup_{i=1}^p L^i(u_0, x_0)\right) = \cup_{i=1}^p L^i(u_t, x_t)$$

In both cases, $\lambda = 0$ implies **degeneracy** in law of $D_x\phi^t$.
Inspiration from Baxendale '89 and other work à là Furstenberg

$$D_{x_0}\phi^t_{u_0,\omega}(L_{(u_0,x_0)}) = L_{(u_t,x_t)}$$



Strong Feller

$$u(t,\cdot):\mathbb{T}^d \to \mathbb{R}^d\,, \quad \dot\phi^t(x) = u(t,\phi^t(x))\,, \quad u_t = u(t,x)\,, \quad x_t = \phi^t(x_0)\,.$$

Definition

Let (z_t) be a Markov process on a Polish space Z. We say it has the strong Feller property if for all bounded measurable $\phi : Z \to \mathbb{R}$, have

$$z \mapsto \mathbf{E}\big(\phi(z_t)|z_0 = z\big)$$

is continuous for all $t > 0$.

Method requires **strong Feller** for $z_t = (u_t, x_t)$ process:

- For finite-dimensional processes: Hörmander's condition.
- In infinite-dimensions: Malliavin calculus with nonadapted controls
  - Necessary to force all sufficiently high Fourier modes in NSE

Almost-sure correlation decay: two-point motion

- Consider the *two*-point motion (u_t, x_t, y_t) with $(x \neq y)$:

$$\partial_t x_t = u_t(x_t)\,, \quad \partial_t y_t = u_t(y_t).$$

  Markov process on $\mathbf{H} \times \{x \neq y\}$

- Basic principle: **Averaged** mixing for (u_t, x_t, y_t) implies **almost-sure** mixing for (x_t):
- Basic idea why: apply Borel-Cantelli after the following L^2 trick (Dolgopyat-Kaloshin-Koralov '04, Ayyer-Liverani-Stenlund '07)

$$\begin{aligned}\mathbb{P} \times \mu\left(\left|\int f \circ \phi^n g dx\right| > e^{-qn}\right) &\leq e^{2qn} \int |\mathbf{E}_{u,x,y} f(x_n) f(y_n) g(x) g(y)|\, dx dy d\mu(u) \\ &= e^{2qn} \int |g(x)g(y)| \cdot |P_n^{(2)} \hat{f}(u,x,y)| dx dy d\mu(u)\end{aligned}$$

  where $P_t^{(2)} \psi(u,x,y) = \mathbf{E}_{(u,x,y)} \psi(u_t, x_t, y_t)$, $\hat{f}(u,x,y) := f(x) f(y)$.

- More quantitative control on constant out front requires regularity of f, g and a more complicated argument.

Averaged correlation decay for (u_t, x_t, y_t)

$$\partial_t x_t = u_t(x_t)\,, \quad \partial_t y_t = u_t(y_t)\,. \quad P_t^{(2)}\psi(u,x,y) = \mathbf{E}_{(u,x,y)}\psi(u_t,x_t,y_t)$$

- Process degenerates near (i) $\|u\|_{H^\sigma} \gg 1$ or (ii) $d(x,y) \ll 1$.
- At best, hope to show $\exists \gamma > 0$, $V = V(u,x,y)$ such that

$$\left| P_t^{(2)}\varphi(u,x,y) - \int_{\mathbb{T}^d \times \mathbb{T}^d} \int_{L^2} \varphi(u,x,y)\mu(du)dxdy \right| \lesssim V(u_0,x_0,y_0)e^{-\gamma t}\|\varphi\|_{L^\infty}\,.$$

  Necessarily, $V(u,x,y) \to \infty$ as $\|u\|_{H^\sigma} \to \infty$ or $d(x,y) \to 0$.
- Harris's Theorem: Irreducibility + Drift condition $P_t^{(2)}V \le Ce^{-\alpha t}V + C'$
- For $d = 2$, can control u via $\hat{V}_{\eta,\beta}(u) = (1 + \|u\|_{H^\sigma}^2)^\beta e^{\eta\|\nabla\times u\|_{L^2}^2}$

$$\partial_t x_t = u_t(x_t)\,, \quad \partial_t y_t = u_t(y_t). \quad P_t^{(2)}\psi(u,x,y) = \mathbf{E}_{(u,x,y)}\psi(u_t,x_t,y_t)$$
$$P_t^{(2)} V \le Ce^{-\alpha t} V + C'$$

To control $d(x,y)$:

- When $d(x,y) \ll 1$, have $|\phi^t(y) - \phi^t(x)| \approx |D_x\phi^t v|\,, \quad v := y - x$
- Morally, positive Lyap exponent should imply exponentially fast repulsion from $\{x = y\}$
- Mathematically: track tangent directions $v_t := D_x\phi^t(v_0)/|D_x\phi^t(v_0)|$. Seek dominant eigenfunction of Feynman-Kac semigroup $\hat{P}_t$

$$\hat{P}_t^q\psi(u,x,v) = \mathbf{E}_{(u,x,v)}|D_x\phi^t(v)|^{-q}\psi(u_t,x_t,v_t)$$
$$= \mathbf{E}_{(u,x,v)}e^{-q\int_0^t \langle v_s, D_{x_s}u_s(v_s)\rangle ds}\psi(u_t,x_t,v_t)$$

on $\psi : \mathbf{H} \times \mathbb{T}^d \times S^{d-1} \to \mathbb{R}$; here $0 < q \ll 1$.

Proposition

For all $q \ll 1$, spectral gap for $\hat{P}_t^q$; dominant eigenvalue $\approx e^{-qt\lambda_1}$, and dominant eigenfunction $\psi_q > 0$.

$$V(u,x,y) = \hat{V}_{\eta,\beta}(u) + d(x,y)^{-q}\psi_q\left(u,x,\frac{y-x}{|y-x|}\right)$$

Concluding remarks

We have initiated a study of Lagrangian chaos and Batchelor-regime passive scalar turbulence!

- Verification of chaotic regimes and consequences (H^1 blowup, H^{-1} decay, L^2 enhanced dissipation, Batchelor's law) for Lagrangian flow for deterministic NSE are largely **intractable**.
- In presence of noise, possible to do much more!

Looking forward:

- Dependence of Lyap. exponent λ on parameters of velocity field process? Reynolds number?
- Ambitious goal: chaotic properties for Eulerian dynamics?
  - Recent progress already for L96 (arXiv:2007.15827), Galerkin NSE is work in progress

Thank you!

1. *Lagrangian chaos and scalar advection in stochastic fluid mechanics,* J Bedrossian, A Blumenthal, S Punshon-Smith arXiv preprint arXiv:1809.06484
2. *Almost-sure exponential mixing of passive scalars by the stochastic Navier-Stokes equations*, J Bedrossian, A Blumenthal, S Punshon-Smith arXiv preprint arXiv:1905.03869
3. *Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection-diffusion by stochastic Navier-Stokes*, J Bedrossian, A Blumenthal, S Punshon-Smith arXiv preprint arXiv:1911.01561
4. *The Batchelor spectrum of passive scalar turbulence in stochastic fluid mechanics*, J Bedrossian, A Blumenthal, S Punshon-Smith arXiv preprint arXiv:1911.11014