Generic regularity of free boundaries for the obstacle problem

Xavier Ros-Oton

Universität Zürich

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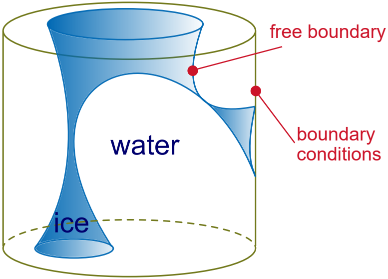

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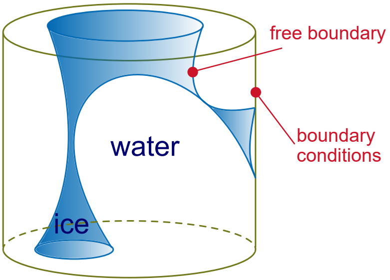

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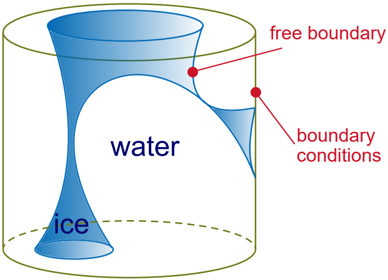

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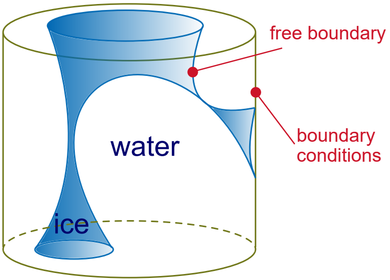
  - $u := \int_0^t \theta \geq 0$ solves:

$$\boxed{u_t - \Delta u = -\chi_{\{u>0\}}}$$



Free boundaries: The obstacle problem

The obstacle problem

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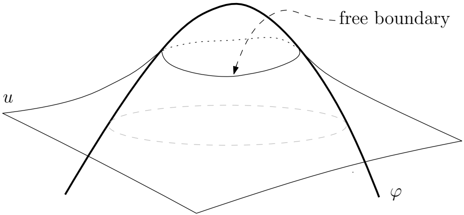
Given $\varphi \in C^\infty$, minimize

$$\mathcal{E}(v) = \int_\Omega |\nabla u|^2 dx$$

with the constraint $\boxed{v \geq \varphi}$



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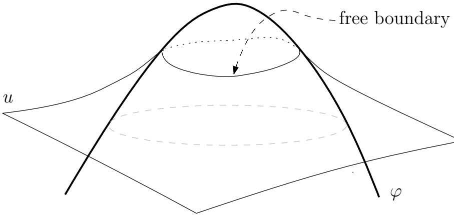

The obstacle problem is

$$\boxed{\begin{cases} v \geq \varphi & \text{in} \quad \Omega \\ \Delta v = 0 & \text{in} \quad \{x \in \Omega \, : \, v > \varphi\} \\ \nabla v = \nabla \varphi & \text{on} \quad \partial\{v > \varphi\}, \end{cases}}$$

(usually with boundary conditions $v = g$ on $\partial\Omega$)

$$\begin{cases} u \geq 0 & \text{in} \quad \Omega, \\ \Delta u = 1 & \text{in} \quad \{x \in \Omega : u > 0\} \\ \nabla u = 0 & \text{on} \quad \partial\{u > 0\}. \end{cases} \quad \longleftrightarrow \quad \begin{array}{c} u \geq 0 \quad \text{in } \Omega \\ \Delta u = \chi_{\{u>0\}} \quad \text{in } \Omega \end{array}$$

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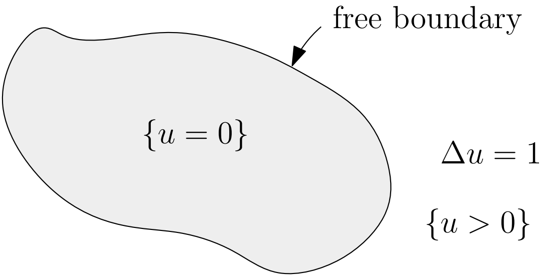

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- Moreover, Stefan problem $\longleftrightarrow$ parabolic obstacle problem !
- Thus, we want to understand better such problem.

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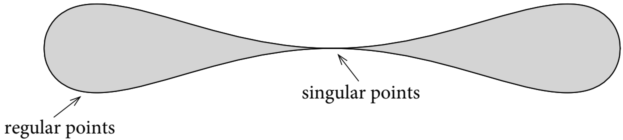

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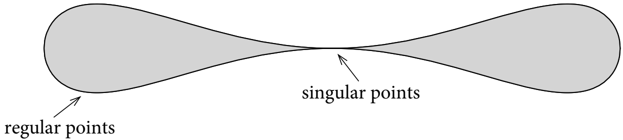

- Furthermore, same results hold for the Stefan problem

To study the regularity of the FB, one considers $\boxed{\text{blow-ups}}$

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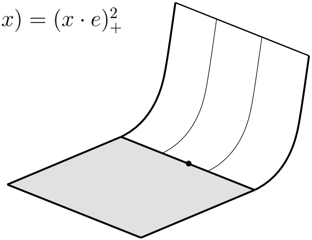

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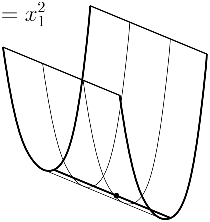

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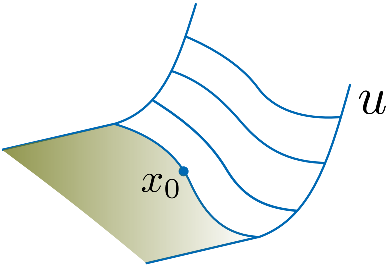

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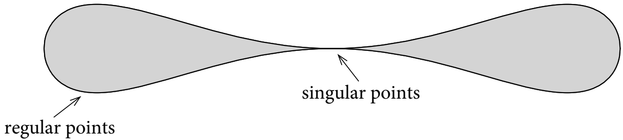

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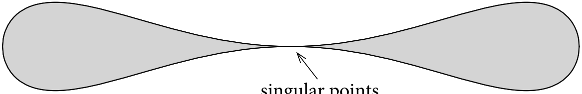

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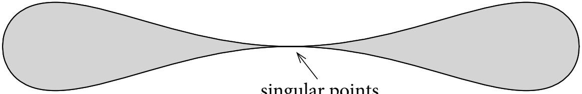

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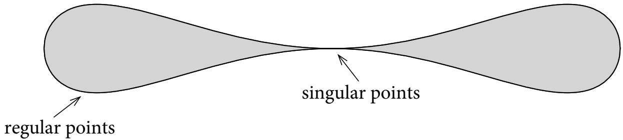

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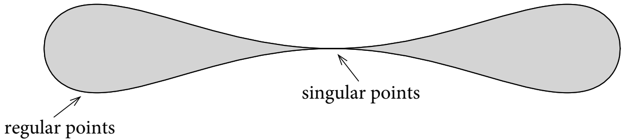

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- Figalli-Serra (2017): Outside a small set of lower dimension, singular points are contained in a $C^{1,1}$ manifold: $\boxed{O(|x-x_0|^3)}$

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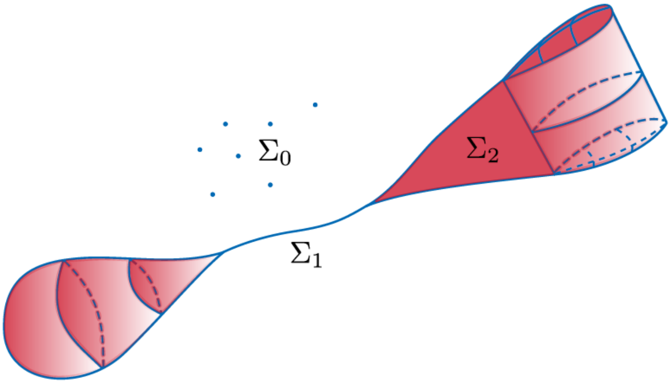
A possible example of a free boundary in $\mathbb{R}^3$ with singularities:



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- Nothing known in higher dimensions!

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Theorem (Figalli-R.-Serra '19)

Let u_λ be the solution to the obstacle problem in $\mathbb{R}^3$, with boundary data $g + \lambda$.

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- In other words: Generically, in $\mathbb{R}^n$, the singular set is very small!

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5) For almost every λ: In lower dimensions, we get no singular points;

   in higher dimensions we get an $(n-4)$-dimensional singular set.

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On the singular set, we improve results of [Weiss '99], and [Figalli-Serra '17].

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- At order $5-\varepsilon$ we have to stop and we cannot control the errors anymore.

GMT Lemma

Lemma

We have a family of sets $E_\lambda \subset \mathbb{R}^n$ *s.t.:*

- *The union* $E = \cup_\lambda E_\lambda$ *has*

$$\dim_{\mathcal{H}}(E) \leq \beta$$

- *For a certain* $k > 0$*, for any* $x_0 \in E_{\lambda_0}$

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Then, for almost every λ*, we have*

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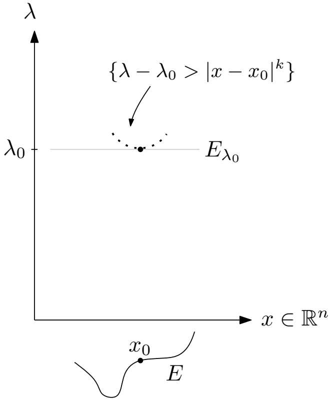

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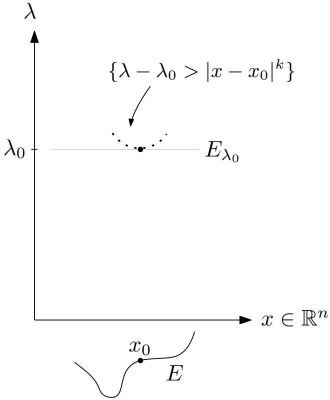

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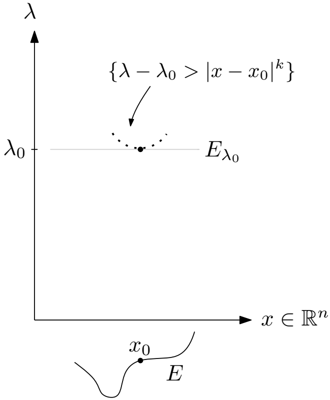

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- We need all dimension-reduction arguments for E, not only for each E_λ !

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Combining this with the previous GMT Lemma, we get the desired result. □

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- Is the $\frac{1}{2}$ sharp? We don't know, but it is *critical* in several ways.

Thank you!