Wild solutions of the Navier-Stokes equations may be smooth for a.e. time

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Summary

1. The Navier-Stokes equations
   - Notions of solutions
   - Partial regularity
   - Main result

2. Convex integration
   - Inductive estimates
   - Gluing step
   - Perturbation step

The system

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Let $\alpha > 0$ and $(-\Delta)^\alpha$ be the differential operator with Fourier symbol $|\xi|^{2\alpha}$. The Navier-Stokes equations with hypo/hyperdissipation on $\mathbb{R}^3 \times [0, +\infty)$ are given by

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = -(-\Delta)^\alpha u \\ \operatorname{div} u = 0 \end{cases} \qquad \text{(NS-}\alpha\text{)}$$

where $(u \cdot \nabla) u := \sum_{i=1}^3 u^i \partial_i u = \sum_{i=1}^3 \partial_i (u^i u) = \operatorname{div}(u \otimes u)$. We consider the Cauchy problem

$$u(\cdot, 0) = u_0 .$$

Taking the divergence of the first equation, we have

$$\Delta p = -\operatorname{div}\operatorname{div}(u \otimes u) .$$

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The (local) energy (in)equality

Multiplying the equation by u,

$$\frac{1}{2}\partial_t|u|^2 + \operatorname{div}\left(u(\frac{|u|^2}{2} + p)\right) = (-\Delta)^\alpha u \cdot u\,.$$

For $\alpha = 1$ this local energy equality reads as

$$\frac{1}{2}\partial_t|u|^2 + \operatorname{div}\left(u(\frac{|u|^2}{2} + p)\right) = \Delta\frac{|u|^2}{2} - |Du|^2\,.$$

Thus we have the global energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|u(x,t)|^2\,\mathrm{d}x = -\int|(-\Delta)^{\alpha/2}u(x,t)|^2\,\mathrm{d}x\,.$$

The natural scaling associated to (NS-α) is given by

$$u \mapsto u_\lambda(x,t) := \lambda^{1-2\alpha}u\left(\frac{x}{\lambda}, \frac{t}{\lambda^{2\alpha}}\right).$$

For $0 \le \alpha < \frac{5}{4}$, the kinetic energy $E(u)(t) := \int|u(x,t)|^2\,\mathrm{d}x$ is supercritical: $E(u_\lambda)(t) = \lambda^{4(\frac{5}{4}-\alpha)}E(u)(t)$.

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Notions of solutions

(i) Distributional solutions: $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \times [0, +\infty))$.

(ii) Leray - Hopf solutions (Leray 1934, Hopf 1951): distributional solutions with global energy inequality for a.e. $t \geq 0$

$$\frac{1}{2}\int |u(t)|^2 \, \mathrm{d}x + \int_0^t \int |(-\Delta)^{\alpha/2} u|^2 \, \mathrm{d}x \, \mathrm{d}\tau \leq \frac{1}{2}\int |u_0|^2 \, \mathrm{d}x \, .$$

Existence was proved by Leray.

(iii) Suitable weak solutions: for $\alpha = 1$, Leray solutions which satisfy the local energy inequality

$$\frac{1}{2}\partial_t |u|^2 + \mathrm{div}\left(u\Big(\frac{|u|^2}{2} + p\Big)\right) \leq \Delta \frac{|u|^2}{2} - |Du|^2 \, .$$

(iv) Classical solutions: blow-up problem.

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The singular set

Let

$$\mathrm{Sing}(u) := \{(x,t) : u \text{ is not locally bounded around } (x,t)\},$$

$$\mathrm{Sing}_T(u) := \{t : \mathrm{Sing}(u) \cap \mathbb{R}^3 \times \{t\} \neq \emptyset\}.$$

Theorem (Leray's estimate on singular times)

Let u be a Leray solution of (NS) in $(0,\infty)$. Then

$$\mathcal{H}^{1/2}(\mathrm{Sing}_T(u)) = 0$$

and $\mathrm{Sing}_T(u)$ is a compact set.

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Leray's short time existence

Lemma

For $u_0 \in H^1$, there exists a unique Leray solution starting from u_0 in $[0, T]$, where $T = \frac{C}{\|\nabla u_0\|^4_{L^2(\mathbb{R}^3)}}$, which is smooth in $(0, T)$.

Indeed, energy estimates on the differentiated equation give

$$\frac{d}{dt}\int |Du|^2\,dx + \int |D^2u|^2\,dx \le \int |Du|^3\,dx.$$

By Hölder and Sobolev inequality

$$\|Du\|^3_{L^3} \le \|Du\|^{3/2}_{L^2}\|D^2u\|^{3/2}_{L^2} \le \|Du\|^6_{L^2} + \frac{1}{4}\|D^2u\|^2_{L^2}.$$

Setting $f(t) := \int |Du|^2\,dx$, it satisfies $f' \le Cf^3$, which implies that the existence time of f is greater than $Cf^{-2}(0)$.

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Proof of Leray's estimate on singular times

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For every $t \in \mathrm{Sing}_T(u)$

$$\int |Du|^2(s,\cdot)\, dx \geq \frac{1}{(t-s)^{1/2}},$$

hence

$$\int_{t-r}^{t+r} \int |Du|^2\, dx\, dt \geq r^{1/2}.$$

Let $\delta > 0$. Extract a Vitali covering $\{(t_i - 5r_i, t_i + 5r_i)\}$ of $\mathrm{Sing}_T(u)$

$$\mathcal{H}^{1/2}_\delta(\mathrm{Sing}_T(u)) \leq \sum_i (5r_i)^{1/2} \leq C \sum_i \int_{t_i-r_i}^{t_i+r_i} \int |Du|^2 \leq \int |u_0|^2$$

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Dimension of the (space-time) singular set

Theorem (Caffarelli, Kohn, Nirenberg '82)

Let u be a suitable weak solution of (NS). Then

$$\mathcal{H}^1(\mathrm{Sing}(u)) = 0.$$

$\mathcal{H}^1$ here is in fact the *parabolic* Hausdorff dimension (covering made by cylinders rather than balls).

- It is based on previous work by Scheffer.
- It recovers Leray's estimate.
- It was recently extended to the hypodissipative range $\alpha \in [3/4, 1)$ in [Tang, Yu '15] and to the hyperdissipative range $\alpha \in (1, 5/4)$ in [Katz and N. Pavlović, '15], [Ozanski, '20], [C., De Lellis, Massaccesi '18].

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Nonuniqueness of weak solutions

Conjecture (Jia, Sverak '15)

Are $L^\infty((0,T);L^{3,\infty}(\mathbb{R}^3))$ weak solutions nonunique?

The conjecture is implied by the fact that the spectrum of a certain linearized operator crosses the imaginary axes. Numerical work by [Guillod, Sverak '17] suggests that this scenario happens.

A.e. smooth wild solutions

Theorem [Buckmaster, C., Vicol '18]

There exists $\beta > 0$ such that the following holds.
For $T > 0$, let $u^{(1)}, u^{(2)}$ be two smooth solutions of (NS) on $[0, T]$.
Then there exists a weak solution u of (NS) such that

- (basic regularity)
  $u \in C^0([0,T]; H^\beta(\mathbb{T}^3)) \qquad \operatorname{curl} u \in C^0([0,T]; L^{1+\beta}(\mathbb{T}^3))$,
- (data) $u \equiv u^{(1)}$ on $[0, \frac{T}{3}]$, and $u \equiv u^{(2)}$ on $[\frac{2T}{3}, T]$,
- (smoothness a.e.) u is smooth in $[0,T] \setminus \Sigma_T$ where $\Sigma_T \subset (0,T]$, is a closed set of times with Hausdorff dimension $< 1 - \beta$.

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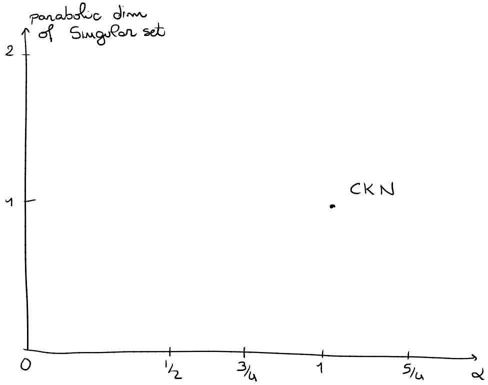
Remarks

- The first part of the statement was obtained by [Buckmaster, Vicol, '18].
- The theorem implies nonuniqueness for any L^2 initial data. Indeed, consider a Leray solution from an L^2 initial datum $u_0 \in L^2(\mathbb{R}^3)$. Wait a little time and find an interval $[T_0, T_1]$ in which it is smooth; define $u^1(t) = u(t + T_0)$. Take u^2 any shear flow (with different initial datum).
- The same proof works for the hypo/hyperdissipative Navier-Stokes equation for any $\alpha \in (0, \frac{5}{4})$. This range was also considered in [Luo, Titi, '18].

Partial regularity and weak solutions at a glance



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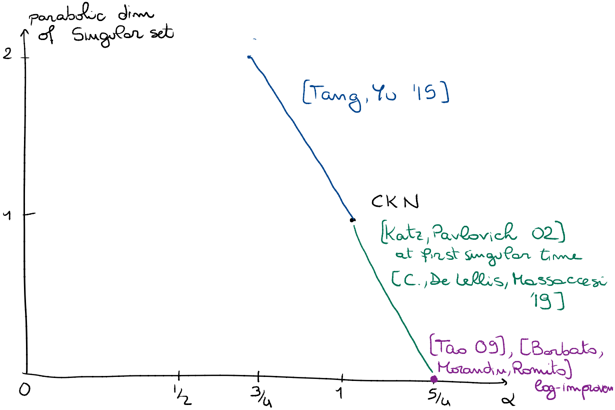

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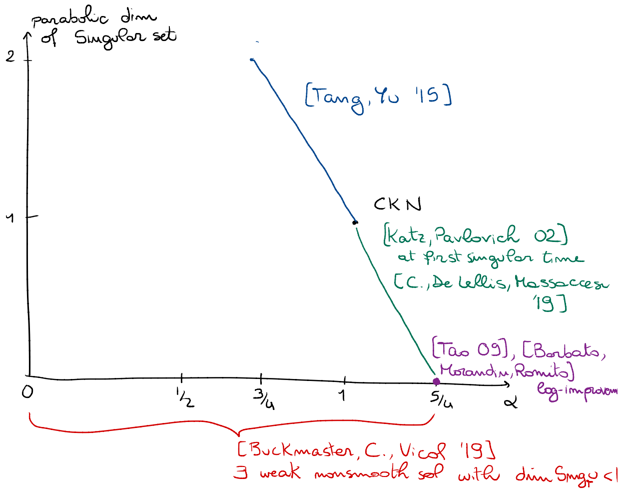

How many bad solutions are there?

How many weak solutions of NS in the space of $C^0([0,T];L^2(\mathbb{T}^3))$ are smooth? Call $X \subseteq Y \subseteq Z$ the sets

$$Z := \{v \in C^0([0,T];L^2(\mathbb{T}^3)) : v \text{ is a weak solution of NS}\}$$

$$X := Z \cap C^\infty([0,T];L^2(\mathbb{T}^3))$$

$$Y := \{v \in Z : v \text{ is smooth on some subinterval of } [0,T]\}.$$

Theorem [C., De Rosa, Sorella, in preparation '20]

The set X is nowhere dense in Z. The set Y is meager in Z.

Nowhere dense sets

A set is called nowhere dense if the interior of its closure is empty.

Convex integration scheme

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We start from (NS) solved with an error (Navier-Stokes-Reynolds system)

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q + (-\Delta)^\alpha v_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0\,, \end{cases}$$

With an inductive procedure we build a sequence (v_q, R_q) such that $v_q \to v$, $R_q \to 0$ as $q \to \infty$.
Size and frequency of our objects are measured by $\delta_q \to 0$ and $\lambda_q \to \infty$, respectively

$$\lambda_{q+1} = \lambda_q^b, \quad b >> 1, \qquad \delta_q := \lambda_q^{-2\beta}.$$

Two steps of the iteration

Gluing step. Replace v_q with $\bar{v}_q$ which enjoys:

- same iterative bounds as v_q
- better properties, e.g. $\bar{v}_q$ is smooth by convolution, or $\bar{v}_q$ is an exact solution on some parts of its domain

It was fully exploited by [Isett '17].

Perturbation step. Build $v_{q+1} = \bar{v}_q + w_{q+1}$ and the stress

$$\begin{aligned}\operatorname{div} R_{q+1} =& \operatorname{div}\left(w_{q+1} \otimes w_{q+1} + \bar{R}_q + \nabla w_{q+1}\right) \\ &+ \partial_t w_{q+1} + \operatorname{div}\left(w_{q+1} \otimes \bar{v}_q + \bar{v}_q \otimes w_{q+1}\right).\end{aligned}$$

w_{q+1} is a combination of stationary solutions at higher frequency than v_q.

Initiated by [De Lellis, Székelyhidi], we follow the scheme of [Buckmaster, Vicol '18] (see also [Modena, Sattig, Székelyhidi '18-'19] and [Brué, C., De Lellis '20]).

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Inductive estimates

- Error is small, of size δ_{q+1}

$$\|\mathring{R}_q\|_{L^1(\mathbb{T}^3)} \leq \lambda_q^{-\varepsilon_R}\delta_{q+1}$$

- v_q is bounded in L^2

$$\|v_q\|_{L^2(\mathbb{T}^3)} \leq C_0 - \delta_q^{\frac{1}{2}}$$

- v_q and R_q live at frequency λ_q

$$\|\mathring{R}_q\|_{H^3(\mathbb{T}^3)}^{1/7} + \|v_q\|_{H^3(\mathbb{T}^3)}^{1/4} \leq \lambda_q$$

All estimates are intended uniform in time.

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Inductive estimates - the set of potential singularities

We perform a Cantor-type construction. Consider the nested bad sets

$$[0,T] \supset [\frac{T}{3}, \frac{2T}{3}] = B_0 \supset \ldots \supset B_q \supset B_{q+1} \supset \ldots$$

- Each B_q is a finite union of intervals and it satisfies

$$\frac{\mathscr{L}^1(B_{q+1})}{\mathscr{L}^1(B_q)} \leq \lambda_q^{-\varepsilon/2}$$

- v_q is an exact solution outside the bad set

$$R_q \equiv 0 \text{ on } [0,T] \setminus B_q$$

- Never modify it again

$$v_{q+1} = v_q \text{ on } [0,T] \setminus B_q$$

$\cap_{q=1}^{\infty} B_q$ is the potential singular set Σ_T. Its dimension is < 1.

Inductive estimates - the set of potential singularities

We perform a Cantor-type construction. Consider the nested bad sets

$$[0, T] \supset [\frac{T}{3}, \frac{2T}{3}] = B_0 \supset \ldots \supset B_q \supset B_{q+1} \supset \ldots$$

- Each B_q is a finite union of intervals and it satisfies

$$\frac{\mathscr{L}^1(B_{q+1})}{\mathscr{L}^1(B_q)} \leq \lambda_q^{-\varepsilon/2}$$

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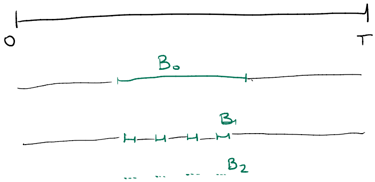

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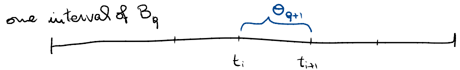
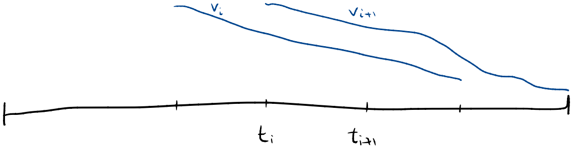
Local smooth solutions



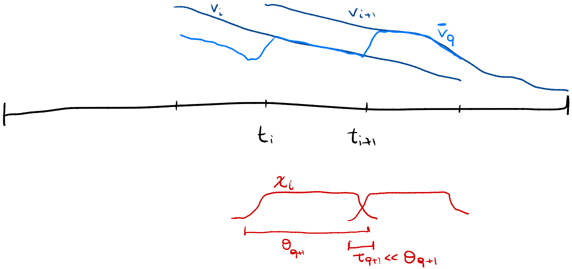
Split any interval in B_q in much smaller intervals of size θ_{q+1}
$t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots$ Take

$$\begin{cases} v_i \text{ solves (NS) in } [t_{i-1}, t_{i+1}] \\ v_i(t_{i-1}) = v_q(t_{i-1}). \end{cases}$$

Let χ_i be a (steep) cutoff in time at scale θ_{q+1} such that its gradient lives at scale $\tau_{q+1} = \lambda_{q+1}^{-\varepsilon/2}\theta_{q+1}$.



Gluing and error estimates



We define the linear interpolation

$$\bar{v}_q = \sum_i \chi_i(t) v_i(x,t)$$

which is an approximate solution with right-hand side given by

$$\begin{aligned} \operatorname{div}(\bar{R}_q) = &\sum_i \partial_t \chi_i (v_i - v_{i+1}) \\ &+ \chi_i(1-\chi_i) \operatorname{div}\left((v_i - v_{i+1}) \otimes (v_i - v_{i+1})\right) \end{aligned}$$

Gluing and error estimates

We invert the divergence and estimate $\|\bar{R}_q\|_{L^1}$:

$$\begin{aligned}\|\mathrm{div}^{-1}(\partial_t \chi_i (v_i - v_{i+1}))\|_{L^1} &\leq \|\partial_t \chi_i\|_{L^\infty} \|\mathrm{div}^{-1}(v_i - v_{i+1})\|_{L^1} \\ &\leq \|\partial_t \chi_i\|_{L^\infty} \|\mathrm{div}^{-1}(v_i - \bar{v}_q)\|_{L^1} \\ &\leq \tau_{q+1}^{-1} \int_{t_{i-1}}^{t_{i+1}} |R_q| \\ &\leq \tau_{q+1}^{-1} \theta_{q+1} \|R_q\|_{L^1} \\ &\leq \lambda_{q+1}^{-\varepsilon/2} \delta_{q+1}.\end{aligned}$$

The second term is better and relies on the choice of θ_{q+1}

$$\|v_i - v_{i+1}\|_{L^2} \leq \theta_{q+1} \|\nabla R_q\|_{L^2} \leq \lambda_{q+1}^{-\varepsilon/4} \delta_{q+1}^{1/2}.$$

Geometric lemma

$v_{q+1} = \bar{v}_q + w_{q+1}$ has to solve

$$\partial_t(\bar{v}_q + w_{q+1}) + \mathrm{div}\,((\bar{v}_q + w_{q+1}) \otimes (\bar{v}_q + w_{q+1})) + \nabla p \\ = \Delta(\bar{v}_q + w_{q+1}) + \mathrm{div}\, R_{q+1}$$

Hence the new stress is

$$\mathrm{div}\, R_{q+1} = \mathrm{div}\,(w_{q+1} \otimes w_{q+1} + \bar{R}_q) \\ + \partial_t w_{q+1} - \Delta w_{q+1} + \mathrm{div}\,(w_{q+1} \otimes \bar{v}_q + \bar{v}_q \otimes w_{q+1}).$$

Lemma

There exist a finite set $\Lambda \subseteq \mathbb{S}^2 \cap \mathbb{Q}^3$ and functions γ_ξ such that for any symmetric matrix $R \in B_{1/2}(Id)$

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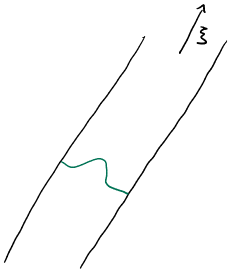
Mikado flows

There is a simple class of exact solutions of Euler in $\mathbb{R}^3$: Mikado flows, introduced by [Daneri, Székelyhidi, '17]. Given a certain direction ξ, for simplicity take $\xi = e_3$ and consider a cutoff $\varphi \in C_c^\infty(\mathbb{R}^2)$

$$W_\xi = W_{e_3} = \varphi(x_1, x_2)e_3.$$

We can suitably rescale and periodize them.



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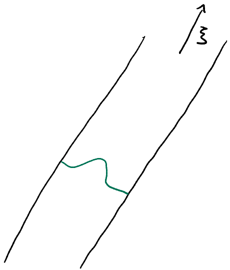
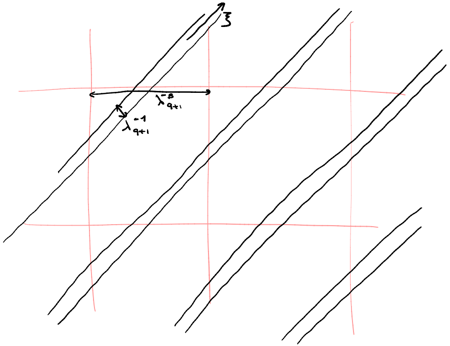


Intermittent Mikado flows?



Given a finite set of directions ξ, W_ξ are stationary solutions of Euler with the following properties

- $\|W_\xi\|_{L^2} = 1$
- W_ξ have mutually disjoint support
- $\int_{\mathbb{T}^3} W_\xi \otimes W_\xi = \xi \otimes \xi$
- W_ξ has frequency λ_{q+1} and small support $|\operatorname{supp} W_\xi| \leq \lambda_{q+1}^{-2(1-\beta)}$.

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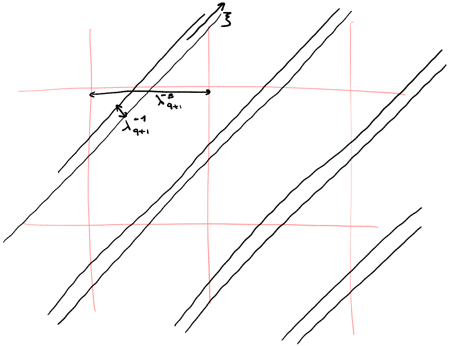

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Perturbation

Wild solutions of the Navier-Stokes equations
Maria Colombo
The Navier-Stokes equations
Notions of solutions
Partial regularity
Main result
Convex integration
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Gluing step
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$$w_{q+1} := \sum_{\xi \in \Lambda} \gamma_\xi (Id - \frac{\bar{R}_q}{\delta_{q+1}}) \delta_{q+1}^{1/2} W_\xi$$

Then looking only at low frequency terms we see that $w_{q+1} \otimes w_{q+1}$ cancels $\bar{R}_q$

$$w_{q+1} \otimes w_{q+1} + \bar{R}_q \approx \sum_{\xi \in \Lambda} \gamma_\xi^2 (Id - \frac{\bar{R}_q}{\delta_{q+1}}) \delta_{q+1} \int_{\mathbb{T}^3} W_\xi \otimes W_\xi$$

New errors: the one coming from the Laplacian

$$
\begin{aligned}
\|\nabla w_{q+1}\|_{L^1} &\leq \sum_\xi \|\gamma_\xi^2 (Id - \frac{\bar{R}_q}{\delta_{q+1}})\delta_{q+1} \nabla W_\xi\|_{L^1} \\
&\leq \sum_\xi \|\gamma_\xi^2 (Id - \frac{\bar{R}_q}{\delta_{q+1}})\delta_{q+1}\|_{L^1} \|\nabla W_\xi\|_{L^1} \\
&\leq \delta_q^{1/2} \lambda_{q+1} |\operatorname{supp} W_\xi|^{1/2}
\end{aligned}
$$

Lemma

Let $p \in \{1,2\}$, $1 < \zeta < \tau$, $N \in \mathbb{N}$ such that $\zeta^{N+4} < \tau^N$. Let $f : \mathbb{T} \to \mathbb{R}$ be such that $\|D^j f\|_{L^p} \leq C_f \zeta^j$ and let g be a $\mathbb{T}/\tau$ periodic function. Then

$$\|fg\|_{L^p} \leq C_f \|g\|_{L^p}$$

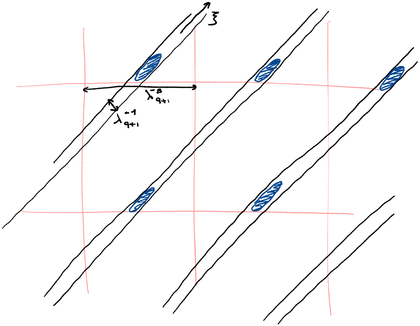
$$\|\mathrm{div}^{-1}(fg)\|_{L^p} \leq C_f \frac{\|g\|_{L^p}}{\tau}.$$

Time oscillations

Unfortunately, Mikado flows do not (barely) satisfy

$$|\operatorname{supp} W_\xi| \leq \lambda_{q+1}^{-2}\delta_{q+1}^{2}.$$

For this reason, [Buckmaster, Vicol '19] introduced time oscillations in W_ξ. In terms of Mikado flows, we consider approximate solutions of Euler "shooting a parcel of fluid" in the Mikado tubes.
[Cheskidov, Luo '20] proposed to use approximate stationary solutions of NS called "viscous eddies".



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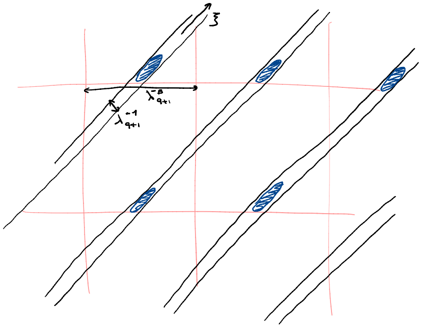

New errors: the one coming from the oscillation

For the oscillation part of the error

$$
\begin{aligned}
\mathrm{div}\, R_{q+1,osc} &= \mathrm{div}\,(w_{q+1} \otimes w_{q+1} + \bar{R}_q) \\
&= \sum_{\xi\in\Lambda} \nabla[\gamma_\xi(Id - \frac{\bar{R}_q}{\delta_{q+1}})\delta_{q+1}^{1/2}]^2\Big(W_\xi \otimes W_\xi + \bar{R}_q\Big) \\
&= \sum_{\xi\in\Lambda} \nabla[\gamma_\xi(Id - \frac{\bar{R}_q}{\delta_{q+1}})\delta_{q+1}^{1/2}]^2\Big(W_\xi \otimes W_\xi - \int W_\xi \otimes W_\xi\Big)
\end{aligned}
$$

Hence

$$
\|R_{q+1,osc}\|_{L^1} \leq \frac{\lambda_q^4}{\lambda_{q+1}}\|W_\xi\|_{L^2} \leq \delta_{q+1}.
$$

The end

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Thank you for your attention!