Instability and non-uniqueness of Leray solutions of the forced Navier-Stokes equations [1]

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[1] Joint with D. Albritton and M. Colombo

The Navier-Stokes system

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = f \\ \operatorname{div} u = 0 \end{cases} \quad \text{on } \mathbb{R}^3 \times [0, T] \qquad \text{(NS)}$$

Regime:

- Continuum description
- Homogeneous (density $\rho = const$)
- Incompressible

Conventional wisdom: In the above regime, the state of the fluid at future times can in principle be predicted from the state of the fluid at the initial time by solving the Navier-Stokes equations.

The Cauchy problem

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = f \\ \operatorname{div} u = 0 \\ u(\cdot, 0) = u_0 \end{cases} \qquad \text{on } \mathbb{R}^3 \times [0, T] \qquad \text{(NS)}$$

- Velocity field: $u : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$
- Pressure: $p : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$
- External body force: $f : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$

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- Given u_0 and f, we look for (u, p)
- We can deduce p from u

$$\Delta p = \operatorname{div}(f - (u \cdot \nabla) u)$$

The energy (in)equality

- Multiplying the equation by u we get

$$(\partial_t - \Delta)\frac{1}{2}|u|^2 + |\nabla u|^2 + \mathrm{div}\left(\left(\frac{1}{2}|u|^2 + p\right) u\right) = f \cdot u \,.$$

- Global energy equality:

$$\frac{1}{2}\int |u(x,t)|^2 \,\mathrm{d}x + \int_0^t \int |\nabla u(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int |u_0(x)|^2 \,\mathrm{d}x + \int_0^t \int f(x,s)\cdot u(x,s)\,\mathrm{d}x\,\mathrm{d}s \,.$$

Energy (in)equality

- Energy:

$$E_u(t) := \frac{1}{2} \int |u(x,t)|^2 \, \mathrm{d}x$$

- Energy dissipation rate:

$$D_u(t) = \int_0^t \int |\nabla u(x,s)|^2 \, \mathrm{d}x \, \mathrm{d}s$$

Assume $f = 0$. Then

$$E_u(t) + D_u(t) = E_u(0) < \infty \quad \forall\, t > 0 \,.$$

Dimensional analysis and strong solutions

- Scaling symmetry:

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)\,, \quad f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)$$

- Dimensional analysis:

$$[x] = L\,, \quad [u] = L^{-1}\,, \quad [t] = L^2 \quad \text{to have } [\Delta u] = [\partial_t u]$$

- Critical spaces have dimensionless norm, e.g. $L_t^\infty L_x^3$
- NS is supercritical: The energy $E_u(t) = \frac{1}{2}\int |u(x,t)|^2\,\mathrm{d}s$ satisfies

$$E_{u_\lambda}(x,t) = \frac{1}{\lambda} E_u(\lambda^2 t)\,.$$

Heuristic explanation

We expect well-posedness when the parabolic regularization beats the non-linearity:

$$(u \cdot \nabla)u << \Delta u \,.$$

- Assume:

$$u \sim A\,, \quad \text{supp } u \sim B_\mu$$

- It holds

$$(u \cdot \nabla)u \sim A^2 \mu^{-1}\,, \quad \Delta u \sim A\mu^{-2}$$

- Critical scaling

$$A^2 \mu^{-1} = A\mu^{-2} \implies A = \mu^{-1} \implies \|u\|_{L^3} = 1$$

Classical solutions

Given u_0 and f smooth and compactly supported, we look for global, smooth solutions to (NS) with bounded energy.

- Well-posedness when u_0 is small
- Well-posedness for small times

Millennium problem

Assume $u_0 \in C_c^\infty$ and $f = 0$. Is there a global smooth solutions to (NS)?

Leray-Hopf solutions

Let $u_0 \in L^2$, $f \in L^1_t L^2_x$.

- [Leray '34], [Hopf '51]: Global solutions to (NS) in the class

$$u \in L^\infty_t L^2_x \cap L^2_t H^1_x \,;$$

- $u(\cdot, 0) = u_0$;
- Energy inequality:

$$\frac{1}{2}\int |u(x,t)|^2 \,\mathrm{d}x + \int_0^t \int |\nabla u(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \\ \leq \frac{1}{2}\int |u_0(x)|^2 \,\mathrm{d}x + \int_0^t \int f(x,s) \cdot u(x,s) \,\mathrm{d}x\,\mathrm{d}s \,.$$

Properties of Leray-Hopf solutions

- Suitability condition:

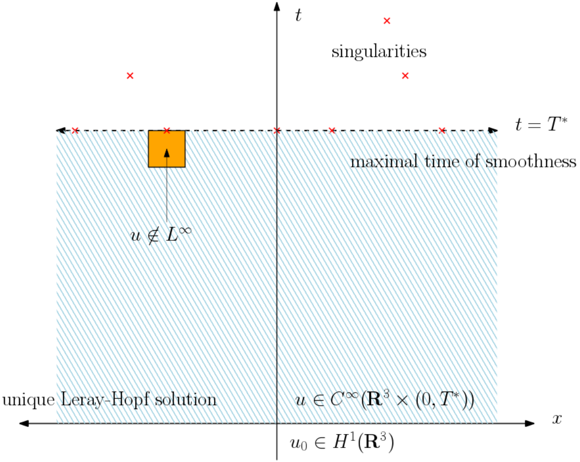
$$(\partial_t - \Delta)\frac{1}{2}|u|^2 + |\nabla u|^2 + \mathrm{div}\left(\left(\frac{1}{2}|u|^2 + p\right)u\right) \leq f \cdot u\,.$$

- Weak-strong uniqueness: Leray-Hopf solutions agree with strong solutions.

- Partial regularity: If $f \in L^{5/2+}_{x,t}$ then

$$\mathcal{P}^1(\text{singular set}) = 0\,, \quad \left(\mathscr{H}^{1/2}(\text{singular times}) = 0\right)\,,$$

[Caffarelli-Kohn-Nirenberg '82].



Weak solutions become relevant if the strong solutions break down.

Uniqueness

The Navier-Stokes equations are used for predicting fluid flows.

- If there is no blow-up, then there is perhaps no need to consider Leray-Hopf solutions.
- If there is blow-up, then the solution may be continued as a suitable Leray-Hopf solution. Is it unique?

Ladyzhenskaya's example

In 1969, Ladyzhenskaya constructed an example of non-uniqueness for (NS) within a Leray-Hopf-type class.

- Self-similarly shrinking domain
- Presence of an external body force
- Non-standard boundary conditions (inhomogeneous, for the stream function)

The example described here can provoke "displeasure" for only one reason. It has been constructed for boundary conditions of type (18) but not for adhesion conditions... The examples presented here are interesting to me in that they refute the entrenched opinion on the "naturalness" for nonstationary problems of physics and mechanics of the class of solutions which have finite energy norm.

Recent progress

- Convex integration method:
  - Non-uniqueness of weak solutions to (NS) in
    $$u \in C_t H_x^{\beta}\,, \quad \text{for some } \beta > 0\,,$$
    [Buckmaster-Vicol '19].
  - Non-uniqueness in $L_t^q L_x^\infty$, $q < 2$ and $C_t L_x^p$, $p < 2$. [Cheskidov-Luo '20].
- The program of Jia, Sverak, and Guillod:
  - Bifurcation from large self-similar solutions.
    Need to prove the existence of an unstable self-similar background [Jia-Sverak '14, '15].
  - Numerical evidence of instability [Guillod-Sverak '17].
- [Albritton-B.-Colombo '21]: Rigorous proof of the prediction of Jia-Sverak-Guillod in the presence of an external body force.

Theorem (Buckmaster, Colombo, Vicol)

There exists $\beta > 0$ *such that the following holds.*
For $T > 0$*, let* $u^{(1)}, u^{(2)}$ *be two smooth solutions of (NS) on* $[0, T]$*. Then there exists a weak solution u of (NS) such that*

- *Regularity:*
$$u \in C^0([0, T]; H^\beta(\mathbb{T}^3))$$
- *Data:*
$$u \equiv u^{(1)} \text{ on } [0, T/3] \text{ and } u \equiv u^{(2)} \text{ on } [2T/3, T]$$
- *Smoothness a.e.* *u is smooth in* $[0, T] \setminus \Sigma_T$ *where* $\Sigma_T \subset (0, T]$*, is a closed set of times with Hausdorff dimension* $< 1 - \beta$*.*

Convex integration in fluid mechanics

Initiated by [De Lellis-Székelyhidi], we follow the scheme of [Bukmaster-Vicol, '18] (see also [Modena, Sattig, Székelyhidi, '20]).

Iterative scheme: We build $(u_q, p_q, R_q)_{q\in\mathbb{N}}$ such that

$$\begin{cases} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q - \Delta u_q = -\operatorname{div} R_q\,, \\ \operatorname{div} u_q = 0, \end{cases}$$

$$u_q \to u \quad \text{in } C^0_t L^2_x\,, \qquad R_q \to 0 \quad \text{in } C^0_t L^1_x\,.$$

Convex integration in fluid mechanics

We make the Ansatz

$$u_{q+1} := u_q + a(R_q) w_\lambda,$$

where

- $\lambda = \lambda_{q+1} \gg \lambda_q$ is the frequency and is much bigger than the typical oscillations in u_q;
- $w_\lambda(x) = w(\lambda x)$, where w is the building block, a high frequency, concentrated vector field.
- $a(R_q)$ is a slow function.

Cancellation of previous error

The new error is obtained as

$$-\operatorname{div} R_{q+1} = \underbrace{\partial_t(a(R_q)w_\lambda) + \operatorname{div}(a(R_q)w_\lambda \otimes u_q + u_q \otimes a(R_q)w_\lambda)}_{\operatorname{div}(L)}$$
$$+ \underbrace{\operatorname{div}(a(R_q)^2 w_\lambda \otimes w_\lambda - R_q)}_{\operatorname{div}(NL)}.$$

We hope that

$$L\,, NL \ll |R_q|\,.$$

- Estimate of L: The space concentration ensure that L is small in L^1
- Estimate of NL: We take w such that the low frequency interaction in $w \otimes w$ is of order 1

$$\int w_\lambda \otimes w_\lambda \, dx = \int w \otimes w \, dx \approx 1$$

and $a(R_q) \approx \sqrt{R_q}$.

A constraint on regularity coming from scaling

The cancellation of the error imposes

$$\|w_\lambda\|_{L^2} = \|w\|_{L^2} \sim 1 \,.$$

The Sobolev inequality implies:

$$\|Dw_\lambda\|_{L^{6/5}} = \lambda \|Dw\|_{L^{6/5}} \geq C\lambda \|w\|_{L^2} \sim \lambda \,.$$

Therefore,

$$\|Du\|_{L^{6/5}} \sim \sum_q \|Dw_q\|_{L^{6/5}} = +\infty \,.$$

No hope to go beyond $\nabla u \in C^0_t L^{6/5}_x$ without changing substantially the scheme.

A result for 2d-Euler

When $d = 2$, the argument above implies that $\nabla u \notin C^0_t L^1_x$.

Theorem (B.- Colombo '21)

There exists a non-trivial solution $u \in C^0([0,1]; L^2(\mathbb{T}^2))$ to (EU) s.t.

- $\omega = \nabla \times u \in C^0([0,1]; L^{1,\infty}(\mathbb{T}^2))$;
- $u(0,\cdot) = 0$.

Moreover, $u \in C^0([0,1]; W^{s,p}(\mathbb{T}^2))$ for any $s \in (0,1)$ and $p \in (1, \frac{2}{1+s})$.

The Jia-Sverak-Guillod picture

Let α_0 be a -1-homogeneous, div-free velocity field. For any $\sigma > 0$ consider a self-similar solution to (NS)

$$\bar{u}_\sigma(x,t) = \frac{1}{\sqrt{t}} \bar{U}_\sigma \left(\frac{x}{\sqrt{t}} \right) \quad \text{with} \quad u_\sigma(x,0) = \sigma \alpha_0$$

- When $\sigma << 1$ existence and uniqueness is guaranteed because the initial condition is small in $L^{3,\infty}$.
- For generic $\sigma > 0$ the existence is guaranteed by [Jia-Sverak '14], uniqueness may fail.

Similarity variables

Let u be a solution to (NS).

- Change of variables: $\xi = x/\sqrt{t}$, $\tau = \log(t) \in (-\infty, T)$

$$u(x,t) = \frac{1}{\sqrt{t}} U(\xi, \tau)\,,$$

- NS in similarity variables: $(\xi, \tau) \in \mathbb{R}^3 \times (-\infty, T)$

$$\partial_\tau U - \frac{1}{2}(1 + \xi \cdot \nabla)U - \Delta U + U \cdot \nabla U + \nabla P = 0$$

For any $\sigma > 0$, $\bar{U}_\sigma$ is a steady state of the (NS) equation in similarity variables.

The Jia-Sverak-Guillod picture

Linearized operator around the steady-state $\bar{U}_\sigma$:

$$-\mathcal{L}_\sigma U = -\frac{1}{2}(1 + \xi \cdot \nabla)U - \Delta U + \mathbb{P}(\bar{U}_\sigma \cdot \nabla U + U \cdot \nabla \bar{U}_\sigma)\,.$$

- $\mathcal{L}_\sigma$ is stable when σ is small enough, i.e.

$$\sigma(\mathcal{L}_\sigma) \subset \{\mathrm{Re}\lambda < 0\}$$

- If $\sigma(\mathcal{L}_\sigma)$ crosses the Imaginary axis at $\sigma = \sigma_0 > 0$ (bifurcation), then we expect non uniqueness for (NS) with initial condition $\sigma\alpha_0$, $\sigma \geq \sigma_0$.

Theorem (Jia-Sverak '15)

Suppose (A) Hopf bifurcation or (B) saddle-node bifurcation. Then, upon truncating properly, there exist two distinct Leray-Hopf solutions with identical compactly supported data u_0, and $|u_0| = O(1/|x|)$ at $x = 0$.

[Guillod-Sverak '17]: Numerical evidence of pitchfork bifurcation.

Main results

Theorem (Albritton-B.-Colombo '21, '22)

Let Ω be $\mathbb{R}^3$, a smooth bounded domain, or $\mathbb{T}^3$. Then, there exists u and $\bar{u}$, two distinct suitable Leray-Hopf solutions to (NS) *with identical body force $f \in L^1_t L^2_x$ and $u(\cdot, 0) = \bar{u}(\cdot, 0) = 0$. When Ω is a bounded domain, u and $\bar{u}$ satisfy no-slip boundary conditions and f is supported far away from the boundary.*

Elements of proof

- Construction of $\bar{U}$, a smooth, unstable vortex ring with compact support in $\mathbb{R}^3$
- Construction of the unstable manifold associated to $\bar{U}$
- We glue nonunique solutions of (NS) on $\mathbb{R}^3$ to solutions on bounded domains and $\mathbb{T}^3$

What's next

- Connection between fluid dynamics instability and non-uniqueness
- Classical stability/instability results
- Vishik's proof of the sharpness of Yudovich class
- Construction of unstable vortex-rings
- Proof of the main Theorem

Thank you for your attention!