On the stationary compressible Navier-Stokes equations

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Weak Solutions à la Leray and Fluid Mechanics – Comments.

In his famous paper published in *Acta Mathematica* in 1934

Sur le mouvement d'un fluide visqueux remplissant l'espace

J. Leray (1906-1998) considered the incompressible homogeneous Navier-Stokes equations

- Constructed a smooth solution up to a characterized existence time T.
- Introduced the concept of weak solution (and for that defined what is called a Sobolev Space).

Weak solutions ("turbulent").

Weak solutions $\Longrightarrow$ few informations on the well posedness of the system

BUT

Such study has real practical interest:

- Its physical meaning (turbulent solutions as named by J. Leray)
- the stability properties of weak solutions on the continuous model help to understand how to construct appropriate stable numerical schemes which do not generally preserve the strong regularity estimates.
- It also helps to better understand the structure of the equations and mathematical properties of important physical quantities.

Weak Solutions à la Leray and Fluid Mechanics – Comments.

The well-known incompressible Navier-Stokes equations

$$\partial_t u + \mathrm{div}(u \otimes u) - \nu \Delta u + \nabla p = 0, \qquad \mathrm{div}\, u = 0.$$

Its stationary version

$$\mathrm{div}(u \otimes u) - \nu \Delta u + \nabla p = 0, \qquad \mathrm{div}\, u = 0.$$

Remarks:

- Stationary version is simpler than nonstationary version.
- Nothing changed if we replace $\nu \Delta u$ by $\nu_x \Delta_x u + \nu_z \partial_z^2 u$ with $\nu_x \neq \nu_z > 0$.

Global weak solutions à la Leray

- A priori estimates on smooth enough solutions
- Approximate the system with better systems allowing:
  1) to construct regular solutions
  and 2) to satisfy the a priori estimates uniformly
- Up to a subsequence, with compactness argument, show the convergence to a weak solution.

For students. See the book by L.C. Evans. *Weak convergence method for nonlinear partial Diff. Eqs.* CBMS, AMS 1990 to learn various methods.

Remark. Usually the last step is the most difficult one to prove global existence of weak solutions: Stability property ! It requires to deeply understand the PDEs under consideration and some of its properties through:
Compactness, defect and young measures characterization, convexity, quasi-convexity, compensated-compactness, concentrated-compactness, maximum principle methods for instance.....

Stability is not always the most difficult part: see for instance the density dependent viscosities case where construction of approximate solutions has been a long story finishing with real insight by J. Li – Z.P. Xin, A. Vasseur - C. Yu, D.B.-A. Vasseur-C. Yu.

Weak Solutions à la Leray and Fluid Mechanics – Comments.

The barotropic compressible Navier-Stokes equations

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla p(\rho) = 0, \qquad \partial_t \rho + \mathrm{div}(\rho u) = 0$$

with $s \mapsto p(s)$ the given pressure law.

Its stationary version

$$\mathrm{div}(\rho u \otimes u) - \nu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla p(\rho) = 0, \qquad \mathrm{div}(\rho u) = 0.$$

Remarks:

- Stationary version is completely different from the nonstationary version.
- Everything changes if we replace $\mu\Delta u$ by $\mu_x \Delta_x u + \mu_z \partial_z^2 u$ with $\mu_x \neq \mu_z > 0$.
- The result depends strongly on the pressure state laws.

Difficulties in the compressible setting

- Weak bound of the divergence of the velocity and transport equation
- The velocity is not a given function and is linked implicitly to the density
- Singular behaviors of the density (oscillation, concentration)
- Possible degeneracy: vacuum state
- Possible non local properties.
- Strongly use the structure of the system.

Comment: In the usual barotropic NS equation $\mathrm{div} u = p(\rho) +$ compact in space.

Existence comment:
Note that for compressible NS equations (and non-homogeneous incompressible NS eqs), there exists several notion of solutions:

- Weak solution à la Leray,
- Intermediate solution à la Hoff,
- Critical solution à la Kato,
- Regular solution à la Matsumura-Nishida

Weak Solutions à la Leray and Fluid Mechanics – Bibliography.

Non-stationary barotropic compressible Navier-Stokes equations.

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathcal{D} + \nabla p(\rho) = 0$$

where $\mathcal{D}$ is the diffusive quantity.

Isotropic constant viscosities and monotone pressure state laws after a fixed value

- P.–L. Lions (CRAS 1993 and Oxford Univ Press 1998)
  $p(\rho) = a\rho^\gamma$ with $\gamma \geq 3d/(d+2)$ and $\mathcal{D} = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u$.
- E. Feireisl, A. Novotny, H. Petzeltova (J. Math Fluid Mech 2001) and E. Feireisl (J. Diff. Eqs 2002)
  $p(\rho) = a\rho^\gamma$ with $\gamma > d/2$ and $\mathcal{D} = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u$.
  or more generally
  $p \in \mathcal{C}^2([0,+\infty))$ with $p(0) = 0$ with $a\rho^{\gamma-1} - b \leq p'(\rho) \leq \frac{1}{a}\rho^{\gamma-1} + b$ (with $a, b > 0$ and $\gamma > d/2$) and $\mathcal{D} = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u$.
- P.I. Plotnikov and W. Veigant (Siam J. Math Anal. 2015).
  $p(\rho) = a\rho^{d/2}$ and $\mathcal{D} = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u$.

Main steps in the proof

- Energy estimates
- Extra integrability on the pressure
- Defect measure propagation related to $\rho \log \rho$ using that

$$\mathrm{div} u^\varepsilon = p(\rho^\varepsilon) + \text{ compact in space}$$

based on renormalized solution namely

$$\partial_t(\rho \log \rho) + \mathrm{div}(\rho \ \log \rho u) + \rho \mathrm{div}\, u = 0$$

and

$$\partial_t(\overline{\rho \log \rho}) + \mathrm{div}(\overline{\rho \log \rho u}) + \overline{\rho \mathrm{div}\, u} = 0 \text{ where } \bar{\cdot} \text{ means weak limit}$$

in the case $\gamma \geq 3d/(d+2)$: Use truncation on the density if $\gamma > d/2$.

Difference + integration in space + Use the monotonicity after a fixed value of the pressure !!.

Does not work for anisotropic viscosities or non monotone pressure laws !!

Weak Solutions à la Leray and Fluid Mechanics – Bibliography.

Anisotropic constant viscosities and non monotone pressure state laws

- D.B. and P.–E. Jabin (*Annal Math*, 2018)

Non monotone pressure law: p $\in \mathcal{C}([0,+\infty))$, p locally Lipschitz on $(0,+\infty)$ with $p(0)=0$

$$C^{-1}\rho^\gamma - C \leq p(\rho) \leq C\rho^\gamma + C$$

and

$$|p'(s)| \leq \overline{p}s^{\gamma-1}$$

with $\overline{p}>0$ and $\gamma \geq 2d/(d+2)$.

Anisotropic viscous term:

$$\mathcal{D} = -\mathrm{div}(A(t)\nabla u) - (\mu+\lambda)\nabla \mathrm{div}\, u$$

where

$$A(t) = \mu \mathrm{Id} + \delta A(t)$$

with $\mu>0$ and $2\mu/3+\lambda-\|\delta A(t)\|_{L^\infty}>0$.
Assuming

$$\gamma > d/2[(1+1/d)+\sqrt{1+1/d^2}$$

there exists $C_\star$ such that if

$$\|\delta A\|_{L^\infty} \leq C_\star(2\mu+\lambda)$$

then OK.

Main steps in the proof

- Energy estimates
- Extra integrability on the pressure
- Show control of
$$\|\rho\|_{p,\theta} = \sup_{h\leq 1/2} |\log h|^{-\theta} \int_{\Omega^2} \frac{|\rho(t,x)-\rho(t,y)|^p}{(h+|x-y|)^d} dx\, dy$$
using first propagation with appropriate weights $w(t,x)+w(t,y)$ and then quantify the effects of the weights
- Use that for any $s>0$, $0<\theta<1$ and $p\in[1,+\infty)$ one has
$$W^{s,p} \subset W^p_{\log,\theta} \subset L^p$$
which are compact.

Eulerian point of view of the Lagrangian estimates by Crippa-De Lellis.

Appropriate non-local quantity to study general stress tensors to treat non-locality on some quantities.

Weak Solutions à la Leray and Fluid Mechanics – Bibliography.

Semi-stationary stokes equations.

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$
$$-\mathcal{D} + \nabla p(\rho) = 0$$

where $\mathcal{D}$ is the diffusive quantity and

$$\Omega = \mathbf{T}^3, \qquad \int_\Omega \rho = M, \qquad \int_\Omega u = 0.$$

- P.–L. Lions (Oxford Univ. Press 1998),
  See also V. Vaigant and A.V. Kahikhov, N. Masmoudi and P. Zhang.
  $p(\rho) = a\rho^\gamma$ with $\gamma > 1$ and $\mathcal{D} = \mu\Delta u + (\lambda + \mu)\nabla \mathrm{div} u$
- D.B. and P.–E. Jabin (Birthday G. Métivier: Springer Verlag 2017).
  Same restrictions in the anisotropic case than in the non-stationary case above.
  Same compactness characterization method with introduction of weights.

Important remark.

Non-stationary transport equation allows to track propagation of informations on the density

- Propagation of defect measures
- Propagation of non-local informations with weights
- .....

Anisotropy and no size restrictions on the viscosities

- D. B., C. Burtea (Annal. IHP 2020).

Assumptions (H) The viscous term is under the form $\mathcal{D} = \mathrm{div}\tau$ with

$$\tau_{ij}(t, x, D(u)) = A_{ijkl}(t, x)[D(u)]_{kl}$$

where $D(u) = (\nabla u + {}^t\nabla u)/2$ with $A_{ijkl} = A_{ijkl}(t, x) \in W^{1,\infty}((0, T) \times \Omega)$.

- $\tau(t, x, D(u)) : \nabla u = \tau(t, x, D(u)) : D(u)$
- $D(u) \mapsto \tau(t, x, D(u)) : D(u)$ to be weakly lower semi-continuous
- There exists $c > 0$ such that $E = \int_\Omega \tau(t, x, D(u)) : \nabla u \geq c \int_\Omega |D(u)|^2$
- The application $\mathcal{A} : v \mapsto -\mathrm{div}\tau(t, x, D(v))$ is a second order invertible operator such that $\mathcal{A}^{-1}\nabla\mathrm{div}$ is a bounded operator from $L^{3/2-}(\Omega)$ into $L^{3/2-}(\Omega)$.

Theorem

Let us assume ρ_0 satisfies

$$\rho_0 \geq 0, \qquad 0 < M = \int_\Omega \rho_0 < +\infty, \qquad \int_\Omega \rho_0^\gamma < +\infty$$

where $\gamma > 1$ and the strain tensor satisfies assumptions (H). Then there exists a global weak solution (ρ, u) of the semi-stationary compressible system with

$$\rho \in \mathcal{C}([0,T]; L^\gamma_{\text{weak}}(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega), \qquad u \in L^2(0,T; H^1(\Omega)) \text{ with } \int_\Omega u = 0.$$

New ideas:

- Strongly use the absence of the term $\rho u \otimes u$
- Define a new defect measure related to the energy.

No restriction on the anisotropy !

- Energy estimates
- Extra integrability on the pressure
- Compactness on the velocity differentiating in time the momentum eqs
- Defect measure control linked to the pressure

$$\partial_t(\overline{\rho^\gamma} - \rho^\gamma) + \mathrm{div}((\overline{\rho^\gamma} - \rho^\gamma)u) + (\gamma - 1)(\overline{\rho^\gamma} - \rho^\gamma)\mathrm{div}\, u + \overline{\nabla_\theta u : \nabla_\theta u} - \nabla_\theta u : \nabla_\theta u = 0.$$

and multiplying by $(\overline{\rho^\gamma} - \rho^\gamma)^{\frac{1}{\gamma}-1}/\gamma$ therefore

$$\partial_t(\overline{\rho^\gamma} - \rho^\gamma)^{\frac{1}{\gamma}} + \mathrm{div}\big((\overline{\rho^\gamma} - \rho^\gamma)^{\frac{1}{\gamma}} u\big) \leq 0.$$

Weak Solutions à la Leray and Fluid Mechanics – bibilography.

Density dependent viscosities.

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - 2\mathrm{div}(\mu(\rho) D(u)) - \nabla(\lambda(\rho)\mathrm{div} u) = 0.$$

Non exhaustive references.

- D. B., B. Desjardins, C.K. Lin (CPDE 2001)
- D. B., B. Desjardins (CMP 2003)
- D. B., B. Desjardins (CRAS 2004)
- A. Mellet, A. Vasseur (CPDE 2007)
- P.B. Mucha, M. Pokorny, E. Zatorska (DCDS 2014)
- D. B., B. Desjardins, E. Zatorska (JMPA 2015)
- A. Vasseur, C. Yu (Inventiones 2016, Arxiv 2015)
- J. Li, Z.P. Xin (ArXiv:1504.06826 2015)
- D. B., A. Vasseur, C. Yu (Submitted JEMS 2019) –>

This is an other framework than Lions-Feireisl framework.

All these works are based on the BD entropy found

- in CPDE 2001 for $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$
- in CRAS 2004 section mécanique for general case with $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$.

Results for the anisotropic steady Navier-Stokes equation

A few results for the isotropic case

The Navier-Stokes system for a steady compressible viscous barotropic fluid

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \Delta_\theta u + \nabla \rho^\gamma = g \end{cases} \qquad (\mathcal{SNS})$$

where $\Delta_\theta = \mu\Delta \cdot + (\mu+\lambda)\nabla \mathrm{div} \cdot + \mu\theta\partial_3^2 \cdot$ and

$$\Omega = \mathbf{T}^3, \qquad \rho \geq 0, \qquad \int_\Omega \rho = M > 0, \qquad \int_\Omega u = 0.$$

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There are a lot of important results such as:

- strong solutions: A. Valli and W. Zajaczkowski '86, M. Padula '87, A. Novotný '94.

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The method proposed by D. B. and P.-E. Jabin seems to not apply in this context!

No time dependency allows to adapt the idea by D.B., C. Burtea (Ann. IHP 2020)
It allows to include in the stationary case the term $\rho u \otimes u$.

Included in D.B, C. Burtea (To appear JMPA 2020).

Theorem

Let $\gamma > 3$ and $M > 0$ be given. Let g be such that

$$g \in (L^{\frac{3(\gamma-1)}{2\gamma-1}}(\Omega))^3, \qquad \int_\Omega g = 0$$

and the constants μ, λ and θ such that

$$\mu, \qquad \mu + \lambda > 0 \text{ and } \theta > -1.$$

There exists a universal constant c_0 such that if

$$(1 + |\theta|)|\theta|\mu \frac{2\lambda + \mu}{(\lambda + \mu)^2} \leq c_0$$

then there exists a pair $(\rho, u) \in L^{3(\gamma-1)}(\Omega) \times (W^{1,\frac{3(\gamma-1)}{\gamma}}(\Omega))^3$ which is a weak-solutions of the stationary Navier-Stokes system.

Important remarks.

- Similar constraint by D.B., P.–E. Jabin for the nonstationary Navier-Stokes equations.
- If the bulk viscosity λ is large enough, anistoropy may be really large.

The anisotropic steady Navier-Stokes equation: Key ideas.

The following identity holds in the steady case:

$$\operatorname{div}\left(\left(\overline{\rho^\gamma} - \rho^\gamma\right) u\right) + (\gamma - 1)\left(\overline{\rho^\gamma} - \rho^\gamma\right) \operatorname{div} u + \overline{\nabla_\theta u : \nabla_\theta u} - \nabla_\theta u : \nabla_\theta u = 0.$$

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which formally gives:

$$\operatorname{div}\left(\left(\overline{\rho^\gamma}-\rho^\gamma\right)^{\frac{1}{\gamma}}u\right)=\frac{1}{\gamma}\left(\overline{\nabla_\theta u:\nabla_\theta u}-\nabla_\theta u:\nabla_\theta u\right)\left(\overline{\rho^\gamma}-\rho^\gamma\right)^{\frac{1}{\gamma}-1}$$

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We have to recover extra integrability properties for ∇u: this is possible when $amma > 3$. Formally,

$$\nabla u \approx \rho^\gamma$$

such that a bound on $\rho^\gamma \in L^{2+}\left(\mathbb{T}^3\right)$ will ensure an uniform bound for $\nabla u \in L^{2+}\left(\mathbb{T}^3\right)$. This requires a smallness condition on the quantity of anisotropy or that λ the bulk viscosity to be large.

Results for the anisotropic steady Navier-Stokes equation

For any $h > 0$

$$\operatorname{div}\left(\left(\overline{\rho^\gamma} - \rho^\gamma + h\right)^{\frac{1}{\gamma}} u\right) + \frac{\gamma - 1}{\gamma}\left(\overline{\nabla_\theta u : \nabla_\theta u} - \nabla_\theta u : \nabla_\theta u\right)\left(\overline{\rho^\gamma} - \rho^\gamma + h\right)^{\frac{1}{\gamma}-1}$$
$$= \frac{\gamma - 1}{\gamma}\left(\overline{\rho^\gamma} - \rho^\gamma + h\right)^{\frac{1}{\gamma}-1} h \operatorname{div} u$$

We use the parameter h to desingularise quantities because we now

$$\overline{\rho^\gamma} - \rho^\gamma \geq 0 \qquad \text{and} \qquad \gamma > 1.$$

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This relation implies that

$$\overline{\nabla_\theta u : \nabla_\theta u} - \nabla_\theta u : \nabla_\theta u = 0$$

and we recover strong convergence of the gradient in $L^2\left(\mathbb{T}^3\right)$. We may use the compactness of the anisotropic effective flux:

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and we recover strong convergence of the gradient in $L^2\left(\mathbb{T}^3\right)$. We may use the compactness of the anisotropic effective flux:

$$\lim_\varepsilon \rho^\varepsilon\left(\operatorname{div}_\theta u^\varepsilon - (\rho^\varepsilon)^\gamma\right) = \rho\left(\operatorname{div}_\theta u - \overline{\rho^\gamma}\right)$$

to obtain that

$$\overline{\rho^\gamma} = \rho^\gamma.$$

Final comments

Final observations on the stationary compressible Navier-Stokes equations

- Does not seem to extend to pressure functions $p(\rho) \neq \rho^{\gamma}$.

Final comments

Final observations on the stationary compressible Navier-Stokes equations

- Does not seem to extend to pressure functions $p(\rho) \neq \rho^{\gamma}$.
- More general diffusion tensors including non-localities may be considered

$$i)\ \tau = A_{ijkl}(x)\left(\partial_k u^{\ell} + \partial_{\ell} u^k\right)$$
$$ii)\ \tau = \mu_{ij} * \left(\partial_j u^i + \partial_i u^j\right) + \lambda_i * \partial_i \operatorname{div} u$$

Application to microfluidics (see Eringen books).

Important applications remark:
To cover general stress tensor for complex flows is a challenging problem for applications:
Biology, geophysics, mechanics, microfluidics, porous media, astrophysics,.....

Important mathematical remark.

- Density sharp interfaces: solutions à la Leray and solutions à la Hoff seems the most appropriate.
- Characterize motion of interfaces: solutions à la Hoff seems the most approproiate. See for instance the works by D. Hoff – M. Perepetlisa and works by D.B., C. Burtea, M. Hillairet, F. Lagoutière, X. Huang on compressible NS versus mixture theory à la Ishii and R. Danchin – P. Mucha, J.Y. Chemin – M. Paicu – P. Zhang for non-homogeneous incompressible NS eqs.

Thank you for your attention!