오전 9:17 5월 15일 금요일 84%
메모

Axisymmetric Euler Equations

- Goals.

- Contents.

1. LWP for Euler
2. LWP & GWP for axisymmetric Euler
3. Illposedness for Euler
4. Singularity formation for Euler

---

1. Euler equations

$$\text{(Euler)} \begin{cases} \partial_t u + u \cdot \nabla u + \underbrace{\nabla p}_{\text{internal}} = 0 \\ \nabla \cdot u = 0 \end{cases}$$

domain : $\mathbb{R}^n$, $n = 2, 3$

u : velocity

오전 9:18 5월 15일 금요일 84%
메모

u : velocity

p : pressure.

Open problem : $\exists u_0 \in C_0^\infty(\mathbb{R}^3)$ which blows up in finite time.

Solved problem (Elgindi).

$\exists u_0 \in C_0^{1,\alpha}(\mathbb{R}^3)$ $\Rightarrow$ blows up in finite time.

- u : solution to Euler

$$(*)\quad \frac{d}{dt}|u|_{H^s} \le \underbrace{|\nabla u|_{L^\infty}}\cdot \underbrace{|u|_{H^s}}.$$

$$\le |u|^2_{H^s}$$

$\rightarrow$ LWP whenever $s > \frac{n}{2}+1$.

$(*)$ holds $|u|_{C^{k,\alpha}}$ whenever $k \ge 1$, $0<\alpha<1$.

- vorticity $\omega = \nabla \times u$

오전 9:22 5월 15일 금요일 84%
메모

• vorticity $\omega = \nabla \times u$.

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u \\ u = \nabla \times (-\Delta)^{-1} \omega \end{cases}$$

$$\nabla u = \underbrace{\nabla \nabla \times (-\Delta)^{-1}}_{\text{singular integral}} \omega$$

→ certain geometric setups..

$$\Rightarrow \boxed{|\nabla u|_{L^\infty} \lesssim |\omega|_{L^\infty} \log\left(|\omega|_{C^\alpha}\right)}$$

Key Lemma

$$|\omega|_{C^\alpha} \simeq |u|_{C^{1,\alpha}}, \quad 0 < \alpha < 1$$

$$\frac{d}{dt} |\omega|_{C^\alpha} \lesssim |\nabla u|_{L^\infty} |\omega|_{C^\alpha}$$

$$\lesssim |\omega|_{L^\infty} |\omega|_{C^\alpha} \log |\omega|_{C^\alpha}$$

$$\rightarrow \frac{d}{dt} \ln\left(1 + |\omega|_{C^\alpha}\right) \lesssim |\omega|_{L^\infty} \ln\left(1 + |\omega|_{C^\alpha}\right)$$

Beale-Kato-Majda :

$$\int_0^\infty |\omega|_{L^\infty} < \infty$$

오전 9:23 5월 15일 금요일 83%
메모

Beale-Kato-Majda :

$$\|u(t)\|_{C^\alpha} \lesssim \ln\left(1+\|\omega_0\|_{C^\alpha}\right) e^{\int_0^t \|\omega\|_{L^\infty}}$$

2D Euler equation case : $n=2$

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

$$\omega \cdot \nabla u \equiv 0$$

$$\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}.$$

↳ solution is global, with

$$\|u(t)\|_{C^\alpha} \lesssim e^{ce^{ct}} \|u_0\|_{C^\alpha}.$$

Kiselev-Sverak. "Key Lemma"

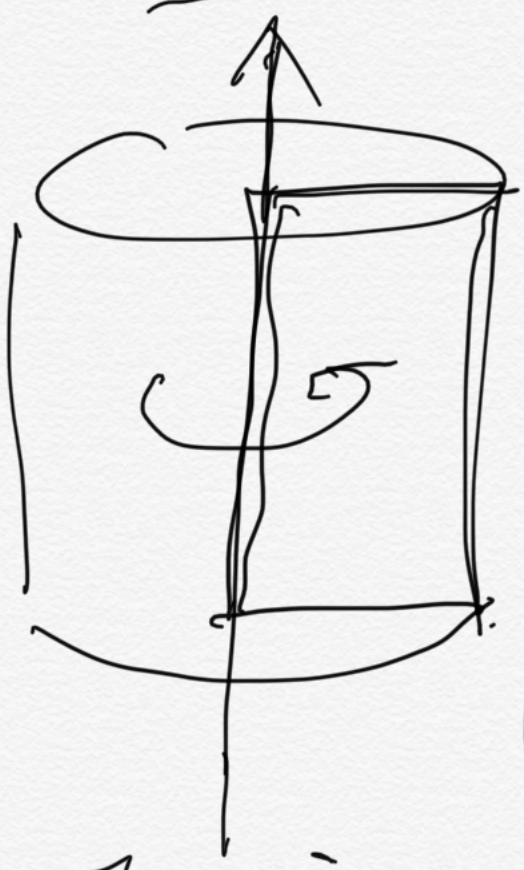
2. Axisymmetric Euler

$\mathbb{R}^3$.

오전 9:23 5월 15일 금요일 83%
메모

2. Axisymmetric Euler

$\mathbb{R}^3$.



$$u = u^r(r,z)\,e^r + u^\theta(r,z)\,e^\theta + u^z(r,z)\,e^z$$

$$\begin{cases} D_t(r\cdot u^\theta) = 0 \\ D_t\left(\dfrac{\omega^\theta}{r}\right) = \dfrac{1}{r^4}\partial_z\left((ru^\theta)^2\right) \\ D_t = \partial_t + u^r\partial_r + u^z\partial_z \\ \partial_r(ru^r) + \partial_z(ru^z) = 0 \\ \partial_z u^r - \partial_r u^z = \omega^\theta \end{cases}$$

(r,z)

$$u^\theta(t=0) = 0 \Rightarrow u^\theta(t) = 0.$$

오전 9:24 5월 15일 금요일 83%
메모

$\Rightarrow$ no-swirl equation

$$D_t\left(\frac{\omega^\theta}{r}\right) = 0.$$

ω^θ may grow.

BKM : $|\omega|_{L^\infty}$ not $\left|\frac{\omega^\theta}{r}\right|_{L^\infty}$.

- GWP for no-swirl : Kernel expression for u^r, u^z

$$(3D)\quad u = \underbrace{\nabla\times(-\Delta)^{-1}}\,\omega^\theta.$$

$$= \frac{1}{4\pi}\int \frac{(x-y)\times\omega(y)}{|x-y|^3}\,dy$$

$$u^r = \int \underline{K^r(r,z,\bar r,\bar z)\,\omega^\theta(\bar r,\bar z)}\,d\bar r\,d\bar z$$

Simple estimate : $\frac{3}{2} > \infty$

$$|u^r| < \int \boxed{\frac{1}{\quad}}\,\frac{\omega^\theta(\bar x)}{\quad}\,d\bar x$$

오전 9:26 5월 15일 금요일 82%
메모

3. ill-posed at C^0

Bourgain-Li → 2D
↘ 3D
=

∂_t $D_t \omega^\theta = \frac{u^r}{r} \omega^\theta$.

ill-posed in $W^{s,p}$ $(s \cdot p = 3)$

for all $s \geq 1$.

---

④ $\left\| \frac{u^r}{r} \right\|_{L^\infty} \cdot \not\lesssim \left\| \omega^\theta \right\|_{W^{s,\frac{3}{s}}}$

$$\lim_{r \to 0} \left(\frac{u^r}{r} \right)(r,0) = (\partial_z u^r)(0,0)$$

$$= C \int \frac{\bar{r}^2 \bar{z}}{(\bar{r}^2 + \bar{z}^2)^{\frac{5}{2}}} \omega^\theta(\bar{r}, \bar{z})$$

$$\omega^\theta(\bar{r}, \bar{z}) := \frac{\bar{r}\bar{z}}{\bar{r}^2 + \bar{z}^2} \left(\ln(\bar{r}^2 + \bar{z}^2) \right)^{-\alpha}$$

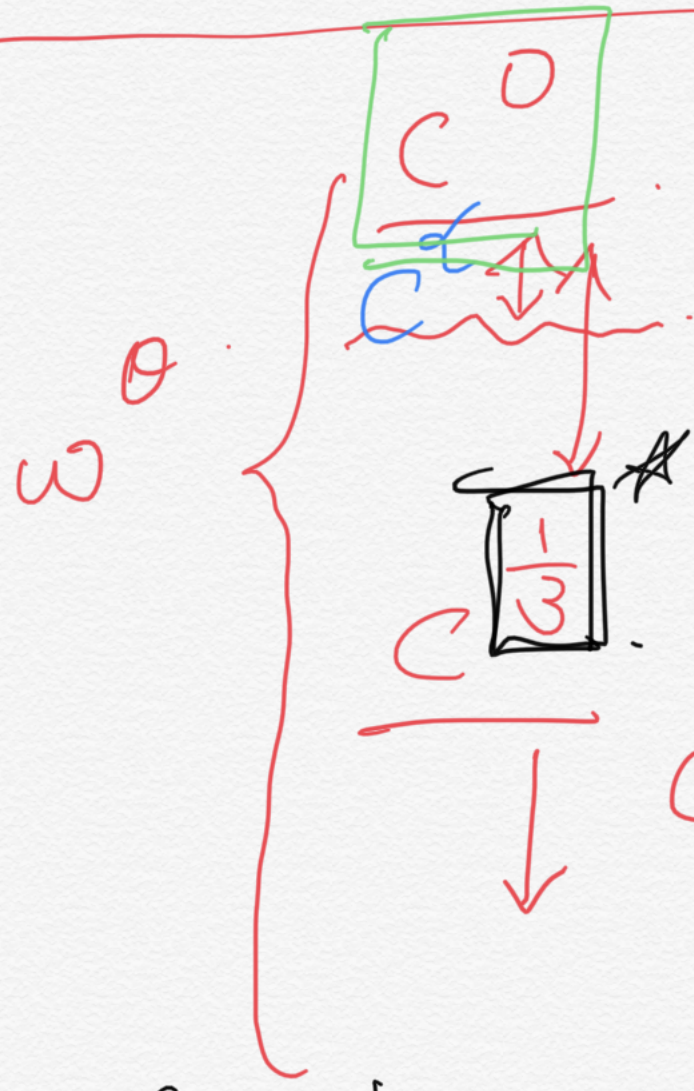
오전 9:26 5월 15일 금요일 83%
메모

Recall LWP in H^s $\left(s > \frac{n}{2}\right)$

$\omega_0^\theta \in H^s$, $s > \frac{n}{2} \Rightarrow$ Sobolev $\frac{\omega_0^\theta}{r} \in L^{3,1}$

* LWP in C^α $(\alpha > 0)$

$\frac{\omega_0^\theta}{r} \in L^{3,1}$ $\Longleftarrow$ $\omega_0^\theta \in C^{\frac{1}{3}+}$



advection VS stretching.

Sarria,

3. illposed at (C^0).

$$\omega^\theta(\bar r, \bar z) := \boxed{\frac{\bar r \bar z}{\bar r^2 + \bar z^2}\left(\ln(\bar r^2 + \bar z^2)\right) \quad r > 0.}$$

$\in C^0$.

polar coordinates (ρ, θ) based on (r, z)

$$\left(\begin{array}{l} r = \rho \cdot \cos\theta \\ z = \rho \sin\theta \end{array}\right)$$

$$= C \int$$

$$= C \cdot \left[\int_0^{\frac{\pi}{2}} \cdots d\theta\right] \underbrace{\int_0^{r_0} \frac{d\rho}{\rho(\ln\rho)^\gamma}}_{\text{not integrable if } \gamma \le 1.}$$

Key Lemma

$$\frac{u^r}{r}(r, z) = 2C \int \boxed{\frac{\bar r^2 \bar z}{(\bar r^2 + \bar z^2)^{\frac{5}{2}}}} \omega^\theta(\bar r, \bar z)$$

$\boxed{\begin{array}{l} r \gtrsim \bar r \\ z \gtrsim \bar z \end{array}}$ $\boxed{+ \text{bdd term}}$

when ω^θ is odd in $\bar z$

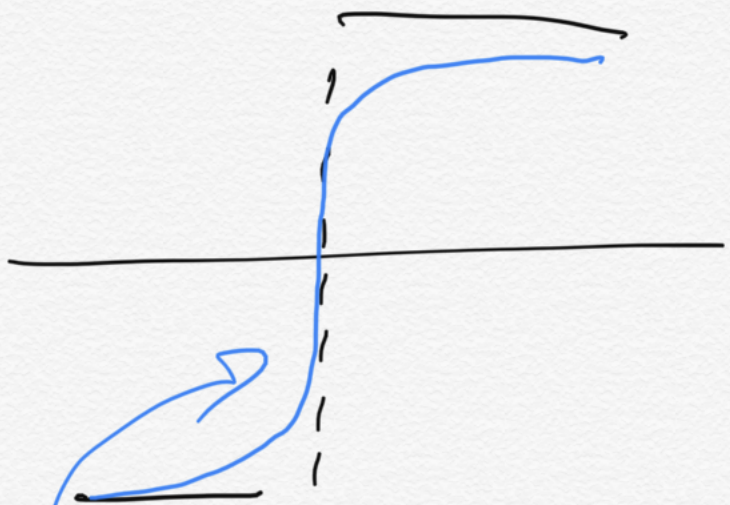
오전 9:27 5월 15일 금요일 82%
메모

Exercise: Hilbert transform in 1D.

$$H[f](x) = PV \frac{?}{\pi} \int \frac{f(y)}{x-y}\, dy$$

If f is odd.

$$(*)\quad H[f](x) = c \int_{2x}^{1} -\frac{f(y)}{y}\, dy + bdd.$$



$$\int_{2x}^{1} \frac{1}{y}\, dy \sim \log x$$

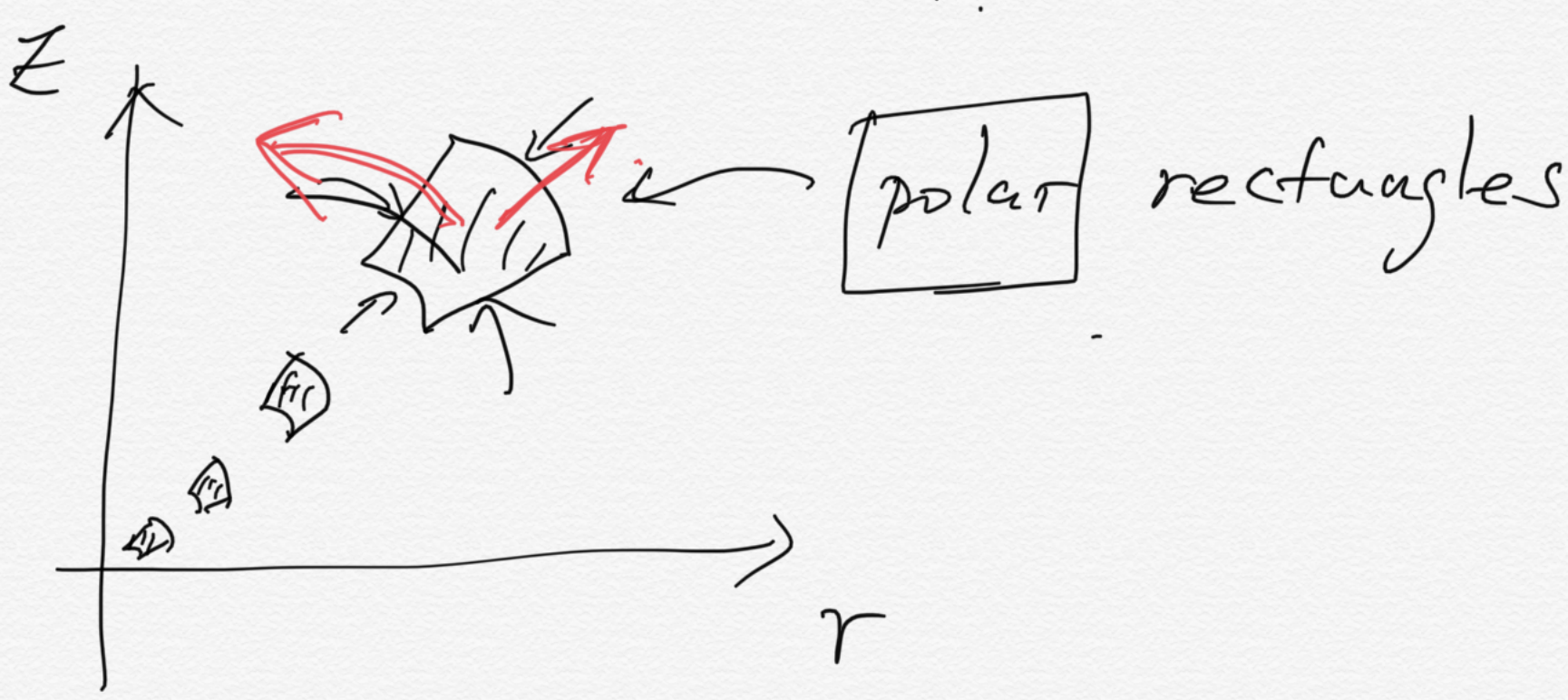
$$f(y) = \underbrace{sgn(y) \cdot |y|^{\alpha}}_{} \cdot (\alpha < 1)$$

$$\int_{2x}^{1} |y|^{\alpha-1}\, dy = \frac{1}{\alpha}(1 - x^{\alpha})$$

$$\Rightarrow \lim_{x \to 0} \frac{Hf(x) - Hf(0)}{|x|^{\alpha}} = \frac{\lim |f(x)|}{\alpha}$$

오전 9:28 5월 15일 금요일 82%
메모

4. Dynamics of C^n Solutions for $0 < \alpha \ll 1$.



$$\underline{u^r} \simeq \iint \boxed{\cos^2\theta \sin\theta} \cdot \omega^\theta(\rho, \theta)\, d\rho\, d\theta.$$

$$\boxed{S}.$$

$(\alpha \to 0)$

$\Rightarrow$ neglect the radial (ρ) velocity.

T. Elgindi:

$$\partial_t \omega + u \cdot \nabla \omega = \frac{u^r}{r} \omega$$

rewrite in polar (ρ, θ)

neglect the radial transport