Global solutions for the relativistic Vlasov-Maxwell with large Maxwell field

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The relativistic Vlasov-Maxwell system (RVM) reads as follows,

$$\begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0, \\ \partial_t E - \mathrm{curl} B = -4\pi j, \quad \mathrm{div} E = 4\pi\rho, \\ \partial_t B + \mathrm{curl} E = 0, \qquad \mathrm{div} B = 0. \end{cases} \tag{1}$$

where $\hat{v} = v/\sqrt{1+|v|^2}$ denotes the relativistic speed and

$$\rho = \int_{\mathbb{R}^3} f(t,x,v)dv, \quad j = \int_{\mathbb{R}^3} \hat{v} f(t,x,v)dv.$$

are the total charge and current densities. The Cauchy problem requires admissible initial data set (E_0, B_0, f_0)

$$\mathrm{div} E_0 = 4\pi\rho_0 = 4\pi \int_{\mathbb{R}^3} f_0(x,v)dv, \quad \mathrm{div} B_0 = 0.$$

Physically speaking, the 3D RVM can be used to describe the evolution of electrons and protons in solar wind under the effect of the electromagnetic forces created by particles themselves.

Continuation criteria.

- Glassey-Strauss (1986) showed that the classical solution can be globally extended as long as the density function has compact support in v for all the time. Klainerman-Staffilani (2002) gave another proof for this result based on a Fourier method.
- Glassey-Strauss (1989) showed that the solution can be extended if the initial density decay at rate $|v|^{-7}$ as $|v| \to \infty$ and $\int_{\mathbb{R}^3} |v| f(t,x,v) dv \leq C$ holds for some constant C.
- Luk-Strain (2016) showed that a regular solution of the Vlasov-Maxwell system can be extended as long as the $\|(1+|v|^2)^{\theta/2} f(t,x,v)\|_{L^q_x L^1_v}$ remains bounded, where $\theta > 2/q,\ 2 < q \leq +\infty$. Extention by Patel (2018).

Small data global solution with asymptotic decay properties

- Glassey-Strauss (1987). This result required compactly supported data x, v and showed that $\rho(t,x) = O(t^{-3})$.
- Schaeffer (2004) allows particles with high velocity but still requires the data to be compactly supported in x.
- Recently, Fajman-Joudioux-Smulevici (2017) proposed a vector field method for relativistic transport equations, based on which Bigorgne (2017 $n \geq 4$, 2018 $n = 3$) removed the compact support assumption on the initial data. Another approach was contributed by Wang (2018) who combined the Fourier analysis with vector field method.

Global large solutions

The large data global regularity is known for
1. 2D relativistic Vlasov-Maxwell: Glassey-Schaeffer(compact support), Luk-Strain (2016).
2. 3D Vlasov-Nordström: Calogero (2006).
3. 3D Vlasov-Poisson: Lion-Perthame (1991), Pfaffelmose(1992), Schaeffer (1992).

However nothing is known for the asymptotic behaviors for the solutions (for the above three models) except the following special large solutions (for 3D RVM):
1. Glassey-Schaeffer (1988) nearly neutral initial data, compact support, can be large on the density but small on the Maxwell field.
2. Rein (1990) large Maxwell field, the initial density with compact support both in x, v.
3. Wang (2020) global solution with symmetry.

Motivation of current work

Motivated by the major open problem: global regularity of RMV, we are specially interested in:

- Find a class of global large solutions with asymptotic decay properties.
- Remove the compact support assumptions based on representation formula.

Statement of the main result.

Theorem (Wei-Y. 2020)

Assume that the initial data set verifies

$$|\nabla^k E_0(x)| + |\nabla^k B_0(x)| \leq M(1+|x|)^{-2-k}, \quad k = 0, 1, 2,$$
$$|f_0(x,v)| \leq \varepsilon_0 (1+|x|+|v|)^{-q}$$

for some constant M and $q > 9$. Then for sufficiently small ε_0, depending only on M and q, the Cauchy problem to RVM (1) admits a global C^1 solution satisfying the following decay properties

$$|E(t,x)| + |B(t,x)| \leq CM(|t-|x||+1)^{-1}(t+|x|+1)^{-1}.$$

for some constant C.

Remarks on the result

1. Similar results hold for Vlasov-Poisson, Vlasov-Nordström.
2. Weaker assumption on Maxwell field

$$|\nabla^k E_0(x)| + |\nabla^k B_0(x)| \leq M(1+|x|)^{-\gamma-k}, \quad \gamma \in (1,2).$$

The assumption in the theorem is just for simplicity.

3. Obviously containing nontrivial charge.
4. Only smallness on the C^0 norm of initial density, the higher order norm can be large.

Characteristic equation for the density function

Now we consider the solution of the Vlasov equation

$$\partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0$$

with the initial condition $f(0, x, v) = f_0(x, v)$ and given Maxwell field (E, B). For fixed $(t, x, v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$, let $X = X(s)$, $V = V(s)$ be the solution of ODE

$$\begin{cases} \partial_s X = \hat{V} \equiv V/\sqrt{1 + |V|^2}, \\ \partial_s V = E(s, X) + \hat{V} \times B(s, X), \\ X(t) = x, \quad V(t) = v, \quad 0 \le s \le t. \end{cases}$$

Then

$$f(t, x, v) = f_0(X(0), V(0)).$$

The representation formula for Maxwell field

Take the electric field E for example as the magnetic field B is similar. The linear solution E^* can be decomposed into $E^* = E_z^* + E_T^* + E_S^*$ with

$$E_T^*(t,x) = -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} f(t-|y-x|,y,v)\frac{dvdy}{|y-x|^2},$$

$$E_S^*(t,x) = -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \nabla_v\left[\frac{\omega+\hat{v}}{1+\hat{v}\cdot\omega}\right]\cdot(E+\hat{v}\times B)f|_{(t-|y-x|,y,v)}\frac{dvdy}{|y-x|},$$

$$E_z^*(t,x) = \mathcal{E}(t,x) - \frac{1}{t}\int_{|y-x|=t}\int_{\mathbb{R}^3}\frac{\omega+\hat{v}}{1+\hat{v}\cdot\omega} f_0(y,v)dvdS_y.$$

Here $\omega = (y-x)/|y-x|$, E_z^* depends only on the initial data.

Iteration process

Let $E^{(0)} = B^{(0)} = 0$. Given $E^{(n-1)}$, $B^{(n-1)}$, define $f^{(n)}$

$$\partial_t f^{(n)} + \hat{v} \cdot \nabla_x f^{(n)} + (E^{(n-1)} + \hat{v} \times B^{(n-1)}) \cdot \nabla_v f^{(n)} = 0, f^{(n)}(0) = f_0.$$

Then let $E^{(n)}$, $B^{(n)}$ be the Maxwell field

$$\partial_t E^{(n)} - \mathrm{curl} B^{(n)} = -4\pi j^{(n)}, \mathrm{div} E^{(n)} = -4\pi \rho^{(n)}, E^{(n)}(0) = E_0,$$
$$\partial_t B^{(n)} + \mathrm{curl} E^{(n)} = 0, \ \mathrm{div} B^{(n)} = 0, B^{(n)}(0) = B_0.$$

Theorem (Glassey-Strauss, 89)

For each $T > 0$, if the kinetic energy density verifies the condition

$$\int_{\mathbb{R}^3} |v| f^{(n)}(t,x,v) dv \le C, \quad \forall 0 \le t \le T, \quad \forall n$$

for some constant $C > 0$, then the above iteration sequence $(E^{(n)}, B^{(n)}, f^{(n)})$ converges to a solution (E, B, f) of RVM.

Weighted norms on Maxwell field

Weighted norms

$$\|K\|_0 = \sup_{x,t}(t-|x|+2)(t+|x|+2)(|E(t,x)|+|B(t,x)|),$$

$$\|K\|_1 = \sup_{x,t} \frac{(t+|x|+2)(t-|x|+2)^2}{\ln(t+|x|+2)}(|\nabla E(t,x)|+|\nabla B(t,x)|),$$

In the work of Rein (1990), $\|K\|_1$ was modified to

$$\|K\|_1 = \sup_{x,t}(t+|x|+2)(t-|x|+2)^\gamma(|\nabla E(t,x)|+|\nabla B(t,x)|),$$

here $\gamma \in (1,2)$, and we assume $|x| \le t+1$.

Key points with compact support

- The density function has uniformly bounded support in v, i.e. $\sup\{|v| | (x, v) \in \mathrm{supp} f(t)\} \leq C_1$.
- Based on the previous fact, the diameter of the support of the density function can be improved

$$\mathrm{diam}\{v | (x, v) \in \mathrm{supp} f(t)\} = O(1/t),$$
$$\mathrm{vol}\{v | (x, v) \in \mathrm{supp} f(t)\} \leq C_2(1 + t)^{-3}$$

Modification of the weighted norm

The $\|\cdot\|_0$ is the same

$$\|K\|_0 = \sup_{x,t}(|t-|x||+1)(t+|x|+1)(|E(t,x)|+|B(t,x)|).$$

This norm is sufficient to close the argument in the exterior region $\{|x|>t\}$. For the estimates in the interior region, we make one of the essential improvements:

$$\|K\|_\gamma = \sup_{|x|\le|y|\le t}(t-|y|+1)^\gamma(t+1)\frac{|E(t,x)-E(t,y)|+|B(t,x)-B(t,y)|}{|x-y|+1}.$$

This norm is weaker than the weighted Lipschitz norm or the weighted C^1 norm. The importance of this norm is that it allows us to avoid the use of ∇f, on which the previous works heavily relied.

Dyadic decomposition of the density function

Take smooth cutoff function $\widetilde{\psi} : \mathbb{R} \to [0, 1]$ supported in $[-2, 2]$. Define $\psi_0(x, v) = \widetilde{\psi}(|(x, v)|)$ and

$$\psi_k(x, v) = \widetilde{\psi}(|(x, v)|/2^k) - \widetilde{\psi}(|(x, v)|/2^{k-1})$$

Then define $f_{0,k} = f_0 \psi_{k-1}$. In particular we have

$$\sum_{k=0}^{+\infty} \psi_k(x, v) = 1, \ \sum_{k=1}^{+\infty} f_{0,k}(x, v) = f_0(x, v),$$
$$\|f_{0,k}\|_0 := \sup_{x,v} |f_{0,k}(x, v)| \leq 2^{(1-k)q} \varepsilon_0.$$

Goal

Denote $f_{[k]} = f_{[k]}^{(n)}$ as the solution of the linear equation

$$\partial_t f_{[k]} + \hat{v} \cdot \nabla_x f_{[k]} + (E + \hat{v} \times B) \cdot \nabla_v f_{[k]} = 0,$$
$$f_{[k]}(0, x, v) = f_{0,k}(x, v).$$

To show that the iteration process converges, we need to show that

- $\sup\{|v| | (x, v) \in \text{supp} f_{[k]}(t)\} \leq C_{1,k}$,
- $\text{vol}\{v | (x, v) \in \text{supp} f_{[k]}(t)\} \leq C_{2,k}(1 + t)^{-3}$.
- $\|K\|_0 \leq \Lambda$, $\|K\|_\gamma \leq \Lambda^2$

Pointwise estimate for the density function

Stating with Maxwell field such that $\|K\|_0 \leq \Lambda$, $\|K\|_\gamma \leq \Lambda^2$, we show that

Proposition (A)

For the density function f as linear solution of Vlasov equation with given E, B, we have

$$f(t,x,v) \leq C(\Lambda)\varepsilon_0(1+|v|)^{-q}.$$

for some constant $C(\Lambda)$ depending only on Λ.

Support for the dyadic piece of the density function

We also need refined estimates for the support of the density function and the charge density.

Proposition (B)

For all $k \geq 1$, $i = 1, 2$, *define*

$$\Lambda_{k,i} = 2^k + (\|K\|_0 \ln(2 + \|K\|_0))^i,$$
$$\Lambda_{k,3} = \ln(1 + \|K\|_\gamma + \|K\|_0) + k + 1.$$

Then for all $t \geq 0$, $x \in \mathbb{R}^3$, *we have*

$$\sup\{|v||(x, v) \in suppf_{[k]}(t)\} \leq C\Lambda_{k,1},$$
$$\int_{\mathbb{R}^3} f_{[k]}(t, x, v)dv \leq C\|f_{0,k}\|_0(2^k + t + |x|)^{-3}\Lambda_{k,1}^5\Lambda_{k,2}^3\Lambda_{k,3}^3.$$

Estimates for the Maxwell field

The convergence of the Maxwell field relies on the following estimate.

Proposition (C)

For sufficiently large Λ, *depending only on* M *(the size of initial Maxwell field), there exists a positive constant* $\varepsilon_* \in (0,1)$ *such that if* $\varepsilon_0 < \Lambda^{-8}(\ln \Lambda)^{-11}$, *then*

$$\|(E^*, B^*)\|_0 \leq \Lambda, \quad \|(E^*, B^*)\|_\gamma \leq \Lambda^2.$$

Proof for Proposition A

Lemma (see also Schaeffer (2004) Lemma 1.1)

Assume that $\|K\| < +\infty$, *and*

$$\frac{dX}{ds} = \hat{V}, \quad \frac{dV}{ds} = E(s, X(s)) + \hat{V}(s) \times B(s, X(s)),$$

for $0 \le s < T^*$. *Then for* $0 \le s \le t < T^*$ *we have*

$$|V(t)| + \|K\|_0 \ln(2 + \|K\|_0) \le C(|V(s)| + \|K\|_0 \ln(2 + \|K\|_0)),$$
$$|V(s)| + \|K\|_0 \ln(2 + \|K\|_0) \le C(|V(t)| + \|K\|_0 \ln(2 + \|K\|_0)).$$

This result means that $(|V(t)| + 1) \sim (|V(0)| + 1)$ along the characteristics, and it implies Proposition (A).

Proof for Proposition B

Based on the following lemmas.

Lemma

Define

$$\Lambda = |V(0)| + (1 + \|K\|_0 \ln(2 + \|K\|_0))^2.$$

Then for $0 \le t < T^*$, *we have*

$$|X(t) - X(0) - t\hat{V}(t)| \le C\Lambda(\ln(1+t) + \ln(1 + |X(0)|)).$$

The key point is to use

$$\frac{d}{ds}(X - s\hat{V}) = -s\frac{d\hat{V}}{ds}, \quad \left|\frac{d\hat{V}}{ds}\right| \le \sqrt{1 - |\hat{V}|^2}(|E| + |B|)$$

This result implies $\mathrm{diam}\{\hat{v} | (x, v) \in \mathrm{supp} f_{[k]}(t)\} = O(\ln t/t)$.

Improved estimates with higher regularity

Lemma

If $|X_i(0)| \le R$, $|V_i(0)| \le R$, $i = 1, 2$ *and* $X_1(t) = X_2(t)$, *then*

$$t|\hat{V}_1(t) - \hat{V}_2(t)| \le C\Lambda(\ln(1 + \|K\|_\gamma + \|K\|_0 + R) + 1),$$

where

$$\Lambda = R + (1 + \|K\|_0 \ln(2 + \|K\|_0))^2$$

Thus we can remove the logarithmic loss using the $\|K\|_\gamma$ norm.

The charge density

The charge density relies on the following lemma of Schaeffer.

Lemma (Schaeffer 2004)

Let $v \in \mathbb{R}^3$ such that $|v| \leq P$, $P \geq 1$. For $\delta > 0$ define the set

$$S = \{w| \; |w| \leq P, \quad |\hat{v} - \hat{w}| \leq \delta\}.$$

Then the Lebesgue measure of S verifies

$$\mu(S) \leq CP^5\delta^3.$$

Proof for Proposition C

Key new estimate

Lemma

For all $p \in [0,2]$ and $t \geq 0$, $x \in \mathbb{R}^3$, we have

$$\int_{|y-x|\leq t} \int_{\mathbb{R}^3} f_{[k]}(t-|y-x|, y, v) \frac{dvdy}{|y-x|^p}$$
$$\leq C\|f_{0,k}\|_0 (2^k + t)^{-p} 2^{(6-2p)k} \Lambda_{k,1}^{2+p} \Lambda_{k,2}^{p} \Lambda_{k,3}^{p}.$$

This lemma relies on the momentum conservation

$$\int_{|y-x|\leq t} \int_{\mathbb{R}^3} (1+\hat{v}\cdot\omega) f_{[k]}(t-|y-x|, y, v) dvdy$$
$$= \int_{|y-x|\leq t} \int_{\mathbb{R}^3} f_{[k]}(0, y, v) dvdy.$$

Decomposition of Maxwell field

We can write

$$E_T^*(t,x) = \sum_{k=1}^{+\infty} E_{T,k}^*(t,x), \quad E_S^*(t,x) = \sum_{k=1}^{+\infty} E_{S,k}^*(t,x), \tag{2}$$

such that

$$E_{T,k}^*(t,x) = -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} f_{[k]}(t-|y-x|,y,v)\frac{dvdy}{|y-x|^2},$$

$$E_{S,k}^*(t,x) = -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \nabla_v\left[\frac{\omega+\hat{v}}{1+\hat{v}\cdot\omega}\right]\cdot(E+\hat{v}\times B)f_{[k]}|_{(t-|y-x|,y,v)}\frac{dvdy}{|y-x|}.$$

Since $f_{[k]}(t,x,v)=0$ for $|x|\ge t+2^{k+1}$, we conclude that $E_{T,k}^*(t,x)=E_{S,k}^*(t,x)=0$ for $|x|\ge t+2^{k+1}$.

Estimate for $\|E_S^*\|_0$

Using the representation formula and the above Lemma, roughly we can show that

$$|E_{S,k}^*(t,x)| \leq \frac{C2^{4k}\Lambda_{k,1}^4\Lambda_{k,2}\Lambda_{k,3}\|K\|_0\|f_{0,k}\|_0}{(|t-|x||+1)(t+|x|+1)}$$
$$\leq \frac{Ck2^{(9-q)k}\Lambda^7(\ln\Lambda)^7\varepsilon_0}{(|t-|x||+1)(|x|+t+1)}.$$

As $q > 9$, summing up we have

$$|E_S^*(t,x)| \leq \frac{C\Lambda^7(\ln\Lambda)^7\varepsilon_0}{(|t-|x||+1)(|x|+t+1)}, \tag{3}$$

which implies $\|(E_S^*, 0)\|_0 \leq C\Lambda^7(\ln\Lambda)^7\varepsilon_0$.

Estimate for $\|E_S^*\|_\gamma$

Improved estimate for E_S^* relies on the improved decay of Maxwell field away from the light cone. For $|x| \leq t$ we have

$$|E_{S,k}^*(t,x)| \leq \frac{C\Lambda_{k,1}^6 \Lambda_{k,2}^3 \Lambda_{k,3}^3 \|K\|_0 \|f_{0,k}\|_0 \ln(t - |x| + 2)}{(t - |x| + 1)^2 (|x| + t + 1)},$$

Hence

$$|E_S^*(t,x)| \leq \frac{C\Lambda^{13} (\ln \Lambda)^{15} \varepsilon_0 \ln(t - |x| + 2)}{(t - |x| + 1)^2 (|x| + t + 1)},$$

Interpolation leads to

$$|E_S^*(t,x)| \leq \frac{C\Lambda^8 (\ln \Lambda)^9 \varepsilon_0}{(t - |x| + 1)^\gamma (t + 1)},$$

which implies $\|(E_S^*, 0)\|_\gamma \leq C\Lambda^8 (\ln \Lambda)^9 \varepsilon_0$.

Estimate of $\|E_T^*\|_0$

Similarly, we can roughly bound that

$$|E_{T,k}^*(t,x)| \leq C2^{2k}\Lambda_{k,1}^5\Lambda_{k,2}^2\Lambda_{k,3}^2(|x|+t+1)^{-2}\|f_{0,k}\|_0$$
$$\leq Ck^2 2^{(9-q)k}\Lambda^9(\ln\Lambda)^{11}(|x|+t+1)^{-2}\varepsilon_0.$$

As $q > 9$, summing up we have

$$|E_T^*(t,x)| \leq C\Lambda^9(\ln\Lambda)^{11}(|x|+t+1)^{-2}\varepsilon_0,$$

which implies $\|(E_T^*, 0)\|_0 \leq C\Lambda^9(\ln\Lambda)^{11}\varepsilon_0$.

Estimate for $\|E_T^*\|_\gamma$

For an interval $I \subset [0, t]$ let

$$E^*_{T,k,I}(t,x) = -\int_{|y-x|\in I}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} f_{[k]} \frac{dvdy}{|y-x|^2}.$$

Then $E^*_{T,k} = E^*_{T,k,[0,1]} + E^*_{T,k,[1,t]}$. The first part decays fast:

$$|E^*_{T,k,[0,1]}(t,x)| \le C\Lambda^6_{k,1}\Lambda^3_{k,2}\Lambda^3_{k,3}t^{-3}\|f_{0,k}\|_0.$$

For $E^*_{T,k,[1,t]}$, we need to use its derivative $\nabla E^*_{T,k,[1,t]}$. By using the Vlasov equation, we have

$$\partial_l E^{*i}_{T,k,I} = A_{w,k,I} + A_{TT,k,I} + A_{TS,k,I}, \tag{4}$$

in which

$$A_{TT,k,I} = \int_{|y-x|\in I} \int_{\mathbb{R}^3} a(\omega, \hat{v}) f_{[k]}(t - |y-x|, y, v) \frac{dvdy}{|y-x|^3},$$

$$A_{w,k,I} = s^{-2} \int_{|y-x|=s} \int_{\mathbb{R}^3} d(\omega, \hat{v}) f_{[k]}(t - s, y, v) dvdS_y \Big|_{s=a}^{s=b},$$

$$A_{TS,k,I} = \int_{|y-x|\in I} \int_{\mathbb{R}^3} d(\omega, \hat{v}) S f_{[k]}(t - |y-x|, y, v) \frac{dvdy}{|y-x|^2}$$
$$= \int_{|y-x|\in I} \int_{\mathbb{R}^3} \nabla_v d(\omega, \hat{v}) \cdot (E + \hat{v} \times B) f_{[k]}|_{(t-|y-x|, y, v)} \frac{dvdy}{|y-x|^2}.$$

The kernels a, d are functions of ω, v, verifying the bound

$$|a(\omega, \hat{v})| + |d(\omega, \hat{v})| \le C(1 + |v|^2)^{3/2}.$$

Bound for $A_{TT,k,[1,t]}$

We split the integral into two parts:

$$\begin{aligned}|A_{TT,k,[t/2,t]}| \leq & Ct^{-3}\Lambda_{k,1}^{3}\int_{|y-x|\leq t}\int_{\mathbb{R}^3} f_{[k]}(t-|y-x|,y,v)dvdy \\ \leq & Ct^{-3}\Lambda_{k,1}^{5}2^{6k}\|f_{0,k}\|_0.\end{aligned}$$

The other part relies on the local charge density decay

$$\begin{aligned}|A_{TT,k,[1,t/2]}| \leq & C\Lambda_{k,1}^{3}\int_{1\leq|y-x|\leq\frac{t}{2}}\|f_{0,k}\|_0(2^k+t)^{-3}\Lambda_{k,1}^{5}\Lambda_{k,2}^{3}\Lambda_{k,3}^{3}\frac{dy}{|y-x|^3} \\ \leq & C\|f_{0,k}\|_0(2^k+t)^{-3}\Lambda_{k,1}^{8}\Lambda_{k,2}^{3}\Lambda_{k,3}^{3}\ln(t).\end{aligned}$$

This shows that

$$\begin{aligned}|A_{TT,k,[1,t]}| \leq & |A_{TT,k,[1,t/2]}|+|A_{TT,k,[t/2,t]}| \\ \leq & C\|f_{0,k}\|_0 t^{-3}\Lambda_{k,1}^{8}\Lambda_{k,2}^{3}\Lambda_{k,3}^{3}\ln(t).\end{aligned}$$

Bound for $A_{TS,k,I}$

$$
\begin{aligned}
& |A_{TS,k,I}(t,x)| \\
\leq\ & C \int_{|y-x|\leq t} \int_{\mathbb{R}^3} (1+|v|^2)^{3/2} (|E|+|B|) f_{[k]}|_{(t-|y-x|,y,v)} \frac{dvdy}{|y-x|^2} \\
\leq\ & C\Lambda_{k,1}^3 \|K\|_0 \int_{|y-x|\leq t} \int_{\mathbb{R}^3} \frac{f_{[k]}(t-|y-x|,y,v)}{t-|y-x|+|y|+1} \frac{dvdy}{|y-x|^2} \\
\leq\ & \frac{C\Lambda_{k,1}^3 \|K\|_0}{t-|x|+1} \int_{|y-x|\leq t} \int_{\mathbb{R}^3} f_{[k]}(t-|y-x|,y,v) \frac{dvdy}{|y-x|^2} \\
\leq\ & \frac{C\Lambda_{k,1}^3 \|K\|_0}{t-|x|+1} 2^{2k} \Lambda_{k,1}^4 \Lambda_{k,2}^2 \Lambda_{k,3}^2 t^{-2} \|f_{0,k}\|_0.
\end{aligned}
$$

Bound for $A_{w,k,[1,t]}$

This is boundary integrals at time 1 and t. At time t:

$$\begin{aligned}
&\left| s^{-2} \int_{|y-x|=s} \int_{\mathbb{R}^3} d(\omega, \hat{v}) f_{[k]}(t-s, y, v) dv dS_y \Big|_{s=t} \right| \\
&\leq C t^{-2} 2^{3k} \int_{|y-x|=t, |y|\leq 2^{k+1}, |v|\leq 2^{k+1}} \|f_{0,k}\|_0 dv dS_y \\
&\leq C t^{-2} 2^{3k} 2^{2k} 2^{3k} \|f_{0,k}\|_0 \leq C t^{-2} 2^{9k} (t - |x| + 1)^{-1} \|f_{0,k}\|_0 .
\end{aligned}$$

Here we may note that $0 \leq t - |x| \leq 2^{k+1}$. At time 1:

$$\begin{aligned}
&\leq C \Lambda_{k,1}^3 \int_{|y-x|=1} \int_{\mathbb{R}^3} f_{[k]}(t-1, y, v) dv dS_y \\
&\leq C \|f_{0,k}\|_0 t^{-3} \Lambda_{k,1}^8 \Lambda_{k,2}^3 \Lambda_{k,3}^3 .
\end{aligned}$$