Multilinear Harmonic Analysis for nonlinear PDEs with potentials

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Outline

- Models and motivation
- The distorted Fourier transform
- The case $d = 1$: Quadratic Klein-Gordon
- The case $d = 3$: Quadratic NLS

Some models

We are going to consider nonlinear dispersive PDEs with potentials such as

- Cubic nonlinear Schrödinger (NLS)

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These are just some examples (for which we have results)

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  Global behavior of solutions, and *scattering/asymptotics* as $t \to \pm\infty$.
- Main motivation is the full *asymptotic stability* of special solutions of nonlinear evolution equations, such as *solitons*, *kinks*...

  where models above appear when linearizing around these solutions.

Motivation and Examples

1. Kink for the ϕ^4 model

$$\partial_t^2 \phi - \partial_x^2 \phi = \phi - \phi^3, \qquad \phi_0(x) = \tanh\left(\frac{x}{\sqrt{2}}\right) \tag{1}$$

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2. Traveling Kinks for defocusing mKdV

$$\partial_t u + \partial_x^3 u = -\partial_x\big(u^3\big), \qquad \phi_{c,x_0}(x+ct) = \sqrt{c}\,\phi_0\big(\sqrt{c}(x+x_0+ct)\big)$$

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3. Vortices for (reduced) 2d Ginzburg-Landau model, with energy

$$E(\phi) = \int_0^\infty \left[\left(\frac{\partial\phi}{\partial t}\right) + \left(\frac{\partial\phi}{\partial\rho}\right)^2 + \frac{N^2}{\rho^2}\phi^2 + \frac{1}{4}(1-\phi^2)^2\right]\rho\, d\rho,$$

$$\phi(\rho) \approx 1 - \frac{N}{2\rho^2} + O\big(\rho^{-4}\big), \qquad \rho \to \infty.$$

- In $\partial_t^2 \phi - \partial_x^2 \phi = \phi - \phi^3$ let $\phi = \phi_0 + v$, small v:

$$\partial_t^2 v - \partial_x^2 v = v - 3\phi_0^2 v - 3\phi_0 v^2 - v^3$$
$$\rightsquigarrow \quad \partial_t^2 v + \big(- \partial_x^2 + 2 + V(x)\big) v = -3\phi_0 v^2 - v^3, \qquad (2)$$

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- Ideal result for these types of equations:

$$\phi = M(\phi_0) + D(t,x) + R(t,x)$$

$M(\phi_0) =$ modulated version of ϕ_0
$D(t,x) =$ is a discrete component (w/ vanishing amplitude)
$R(t,x) =$ is a dispersing radiation component

Other notions of stability

In the example above one has the conserved energy

$$E(t) = \int_{\mathbb{R}} (\partial_t \phi)^2 + (\partial_x \phi)^2 + \frac{1}{2}(1 - |\phi|^2)^2 \, dx.$$

Look at $\phi = \phi_0 + v_1$, $\partial_t \phi = v_2$, $v = (v_1, v_2)$, $\|v(0)\|_{H^1 \times L^2} \leq \varepsilon$

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- *Orbital Stability* (in the Energy space):

$$\sup_{t \in \mathbb{R}} \inf_{x_0 \in \mathbb{R}} \left\| \phi(t, \cdot + x_0) - \phi_0 \right\|_{H^1(\mathbb{R})} \leq \varepsilon.$$

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- *Local Asymptotic Stability* (in the Energy space): for any bounded interval I

$$\lim_{t \to \pm\infty} \|v(t)\|_{H^1 \times L^2(I)} = 0.$$

  Due to Martel-Munoz-Kowalczyk ('15).

General difficulties and methods

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  $\longrightarrow$ expect nonlinear structures to play key role, and nonlinear asymptotic phenomena

- In the free/flat $V = 0$ case, can try to use "classical" methods, e.g., vectorfields, normal forms theory, multilinear estimates ...

- Several issues in the perturbed/distorted $V \neq 0$ case:
  - *Lack of invariance* (and commuting vectorfields)
  - *No conservation of momentum* $\rightarrow$ no standard Fourier analysis
  - More nonlinear and *coherent/resonant interactions* ...

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- Want to use the **distorted Fourier transform** (FT adapted to the Schrödinger operator $H = -\Delta + V$) and develop **multilinear harmonic analysis** in distorted frequency space.

- Fourier-based techniques have been successful in the $V = 0$ case

  (see for example Germain-Masmoudi-Shatah, Gustafson-Nakanishi-Tsai, Ionescu-Pausader, Ionescu-P. ...)

Distorted Fourier Transform (dFT)

- Look at solutions ψ of

$$-\Delta\psi + V(x)\psi = \mu\,\psi.$$

- For $\mu = |k|^2$ have bounded solutions $\psi(x,k) \approx e^{ix\cdot k}$ as $|x| \to \infty$
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- Can also have eigenfunctions $\varphi_j(x)$ corresponding to negative eigenvalues $\mu_1, \dots, \mu_N < 0$
- The distorted Fourier transform is given by

$$\widetilde{\mathcal{F}}(f)(k) := \widetilde{f}(k) := c_d \int_{\mathbb{R}^d} \overline{\psi(x,k)} f(x)\,dx, \qquad \widetilde{f}_j := \int_{\mathbb{R}^d} \overline{\varphi_j}(x) f(x)\,dx$$

$$f(x) = \int_{\mathbb{R}^d} \psi(x,k)\widetilde{f}(k)\,dk + \sum_j \widetilde{f}_j \varphi_j(x)$$

(Weyl-Kodaira-Titchmarsh (theory), Ikebe, Alsholm-Schmidt, Agmon (SR potentials) . . .)

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- Wave operators are

$$W_+ = s - \lim_{t\to\infty} e^{it(-\Delta+V)}e^{it\Delta} = \widetilde{\mathcal{F}}^{-1}\widehat{\mathcal{F}}.$$

  H_V and H_0 are unitary equivalent through wave operators
- Spectral Theory and Wave operators:

  $d = 1$: Deift-Trubowitz ('80s), Weder ($\sim$ '00) ...

  $d \geq 2$: Agmon, Kato, Kuroda ($\leq$ '80s), Yajima ('90s), Weder ('03), Beceanu-Schlag ('10s) ...

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- In the free case $\mu(k,\ell,m) := \delta(k-\ell-m)$.

  For $V \neq 0$ do not have "conservation of momentum" $k-\ell-m=0$.

NSD and oscillations

$$\widetilde{f}(T,k) - \widetilde{f}(t,0)$$
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- Want to develop all necessary harmonic analysis tools.

Quadratic Klein-Gordon (w/ Pierre Germain)

Theorem

Consider the equation

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- *V regular and decaying (say, Schwartz) and* $a(x) \overset{x\to\pm\infty}{\longrightarrow} \ell_\pm$
- *Spectral assumptions:*

$$H_V := -\partial_x^2 + V \quad \textit{has no bound states}$$

and, for all $t \in \mathbb{R}$

$$\widetilde{u}(0, t) = 0$$

(will explain this)

Theorem

Consider initial data $(u(0,x), u_t(0,x)) = (u_0(x), u_1(x))$ *with*

$$\|(\langle\partial_x\rangle u_0, u_1)\|_{H^4} + \|\langle x\rangle(\langle\partial_x\rangle u_0, u_1)\|_{H^2} = \varepsilon \ll 1.$$

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Then:

- *(Global bounds and decay) There exists a unique global solution s.t.*

$$\|u(t)\|_{H^5} \lesssim \varepsilon\langle t\rangle^{C\varepsilon}, \qquad \|u(t)\|_{L^\infty} \lesssim \varepsilon\langle t\rangle^{-1/2}.$$

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$$\|u(t)\|_{H^5} \lesssim \varepsilon\langle t\rangle^{C\varepsilon}, \qquad \|u(t)\|_{L^\infty} \lesssim \varepsilon\langle t\rangle^{-1/2}.$$

- *(Asymptotic behavior) There exists* $W \in L^\infty$ *s.t.*

$$\left\|\widetilde{f}(t,k) - W(k)\exp\left(ic|W(k)|^2\log t\right)\right\|_{L^\infty_k} \overset{t\to\infty}{\longrightarrow} 0$$

where $\widetilde{f}(t,k) \approx e^{it\langle k\rangle}(\partial_t + i\langle k\rangle)\widetilde{u}(t,k)$.

Similar as $t \to -\infty$*, up to conjugating by the scattering matrix of* V*.*

Comments

- $\partial_t^2 u + (-\partial_x^2 + 1 + V)u = a(x)u^2 + \cdots$ is a *true quadratic model*:

  No localization of the nonlinearity; e.g. $a \sim 1$ or $a \sim \mathrm{sign}(x)$ as $|x| \to \infty$

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- If V is non-generic: to ensure $\widetilde{u}(0) = 0$ assume structure on equation: for example, V, a and u_0 are s.t. u is odd/even for f_+ even/odd.

Comments

- $\partial_t^2 u + (-\partial_x^2 + 1 + V)u = a(x)u^2 + \cdots$ is a *true quadratic model*:

  No localization of the nonlinearity; e.g. $a \sim 1$ or $a \sim \mathrm{sign}(x)$ as $|x| \to \infty$

  No transformation to cubic nonlinearity (Yes if $V = 0$, $a = 1$)

- About the $\widetilde{f}(0) = 0$ assumption: let f_+ be the zero energy 'eigenfunction' solution to $H_V f_+ = 0$, $f_+ \overset{x\to\infty}{\sim} 1$.

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- Generic and symmetric cases are easier than the general case.

Application 1: "Continuous subsystem" of ϕ^4

Recall linearization around the kink ϕ_0 of the ϕ^4 equation:

$$\partial_t^2 v + \big(-\partial_x^2 + 2 + V(x) \big) v = -3\phi_0 v^2 - v^3$$

- $V = 3(\phi_0^2 - 1)$ has discrete spectrum: translation mode, an *even* resonance, and internal mode:

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Applications 2

Consider the following equations:

$$\partial_t^2 v + \big(-\partial_x^2 + 1 + V_\ell(x) \big) v = N(v), \quad V_\ell := -\ell(\ell+1)\cosh^{-2}(x) \qquad (\mathrm{KG}_\ell)$$

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- $\ell = 2$ comes from the ϕ^4 equation.
- $\ell = 3$ comes from *quadratic NLKG*

$$\phi_{tt} - \phi_{xx} + \phi = \phi^2, \qquad Q(x) = \frac{3}{2}\cosh^{-2}\left(\frac{x}{2}\right).$$

  Our result gives: *asymptotic stability for even solutions of the associated "continuous subsystem"* $u_{tt} + (-\partial_{xx} + 1 + V_3)u = u^2, \;\; u = P_c u.$

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  - Byproduct of our theorem: any a and b under odd symmetry.

Results on Kinks

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- Donninger-Krieger ('16): $\partial_t^2 u - \partial_x^2 u + V(x)u = 0$, $V \approx -\frac{1}{4}|x|^{-2}$, using distorted Fourier Transform & Vectorfields.

  Donninger-Krieger-Szeftel-Wong ('16): Stability of the catenoid for vanishing mean-curvature flow in Minkowski space.

Results for 1d NLS with potential

$$i\partial_t u + (-\partial_{xx} + V(x))u = \pm|u|^2 u$$

- Case $V = 0$: Deift-Zhou ('90s), Hayashi-Naumkin ('97), Lindblad-Soffer ('06), Kato-P. ('11), Ifrim-Tataru ('15) ...

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- Gong-P. ('19): Generic and non-generic w/ vanishing at zero frequency

  Only require weighted L^1-potential (e.g. barrier)

  Simplified proof: using basic PDO bounds, structure of Jost functions, approximate commutation identity.

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$$\left\| e^{itL} f \right\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{|t|}} \|f\|_{L^1_x}, \qquad L = \sqrt{-\partial_{xx} + V + m^2}$$

$$\left\| e^{itL} f \right\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{|t|}} \|\widetilde{f}\|_{L^\infty_\xi} + \frac{1}{|t|^{3/4}} \|\partial_k \widetilde{f}\|_{L^2_\xi}, \tag{D}$$

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- Most of the 1d theory revolves around controlling the norms in (D). Note

$$\|x f\|_{L^2} \approx \|\partial_k \widetilde{f}\|_{L^2}.$$

dFT in $d = 1$ and strategy

- Let $f_{\pm}(x, k)$ be *Jost functions*:

$$(-\partial_x^2 + V)f_{\pm} = k^2 f_{\pm}, \qquad f_{\pm}(x, k) \approx e^{\pm ixk} \quad \text{as} \quad x \to \pm\infty.$$

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- Denote $T(k)$ (*transmission*) and $R_\pm(k)$ (*reflection*) coefficients s. t.

$$T(k)f_+(x,k) \approx e^{ikx} + R_-(k)e^{-ikx}, \quad \text{as } x \to -\infty,$$
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- Defining

$$\psi(x,k) := \frac{1}{\sqrt{2\pi}} \begin{cases} T(k)f_+(x,k) & \text{for } k \geq 0 \\ T(-k)f_-(x,-k) & \text{for } k < 0, \end{cases}$$

the distorted Fourier transform is

$$\widetilde{\mathcal{F}}(\phi)(k) = \widetilde{\phi}(k) = \int_{\mathbb{R}} \overline{\psi(x,k)} f(x)\,\mathrm{d}x, \qquad f(x) = \int_{\mathbb{R}} \psi(x,k)\widetilde{\phi}(k)\,\mathrm{d}k.$$

Decompositions of ψ and μ

- First, decompose each ψ as

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- Accordingly, decompose the NSD

$$\mu(k,\ell,m) = \int \overline{\psi(x,k)}\psi(x,\ell)\psi(x,m)\,\mathrm{d}x = \mu_S + \mu_R$$

where

$$\mu_S(k,\ell,m) \quad \approx \quad \delta_0(p) \quad + \quad \mathrm{p.v.}\frac{1}{p}, \qquad\qquad p := \beta k + \gamma\ell + \delta m,$$

with $\beta,\gamma,\delta = \pm 1$, and $\mu_R(k,\ell,m)$ a 'smooth' function.

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- Let us concentrate on the $\mathrm{p.v.}$ contribution.

- Set $\qquad \widetilde{f}(t,k) := e^{it\langle k\rangle}\widetilde{v}(t,k), \qquad \widetilde{v} = (\partial_t - i\langle k\rangle)\widetilde{u}.$

- The contribution from $\mathrm{p.v.}\dfrac{1}{k-\ell+m}$ to Duhamel's formula is

$$\approx \int_0^t \iint e^{is\Phi(k,\ell,p)}\,\widetilde{f}(\ell)\widetilde{f}(k-\ell-p)\,\mathrm{p.v.}\frac{1}{p}\,\mathrm{d}\ell\mathrm{d}p\,\mathrm{d}s,$$

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- For $|p| \ll 1$, the $\mathrm{p.v.}$ is singular, but Φ is close to the "flat" one, hence lower bounded (uniformly, for bounded k and ℓ)

$$|\Phi(k,\ell,p)| \gtrsim |\Phi(k,\ell,0)| + O(p) \gtrsim 1$$

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- For $|p| \gtrsim 1$, the $\mathrm{p.v.}$ is smooth, but frequencies in Φ are "uncorrelated" ...

- For $|p| \gtrsim 1$ nonlinear terms look like

$$N(t,k) \approx \int_0^t \iint e^{is\Phi(k,\ell,m)}\, \widetilde{f}(\ell)\widetilde{f}(m)\, \chi(k,\ell,m)\, \mathrm{d}\ell \mathrm{d}m\, \mathrm{d}s,$$

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For $r := |\langle k\rangle - 2| \approx ||\xi| - \sqrt{3}|$ get:

$$N(t,k) \approx \int_{|s|\approx r^{-1}} e^{i\,s\,r}\, s^{-3/2+2\delta}\,\mathrm{d}s \approx r^{1/2-2\delta}$$

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$$\|\partial_k N(t,k)\|_{L^2(r\approx t^{-1})} \gtrsim |t|^{2\delta} \qquad \ldots \quad \text{worse than initial assumption}$$

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  - Dictates the functional framework.
  - First time such a degeneracy is treated in 1d.

    In 2d: Deng-Ionescu-Pausader (EM) and DIPP (gravity-capillary WW).
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- To estimate $\partial_k N$:

  Dyadically decompose according to the size of frequencies,
  the distance from 0 and $\pm\sqrt{3}$, the size of $\Phi = -\langle k\rangle + \langle \ell\rangle + \langle m\rangle$ ...

  Integration by parts, time averaging/normal forms, multilinear estimates...

Case $d = 3$

Theorem (P.-Soffer '20)

Consider

$$i\partial_t u + (-\Delta + V)u = u^2 \tag{NLS}$$

for regular and decaying V and small initial data $u_0 \in H^N \cap L^2(\langle x \rangle^4)$

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Assume $-\Delta + V$ *has no bound states.*

Then, there exists a unique global solution to (NLS) *such that*

$$\|u(t)\|_{L^\infty} \lesssim \langle t \rangle^{-1-\alpha}.$$

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for regular and decaying V and small initial data $u_0 \in H^N \cap L^2(\langle x \rangle^4)$

Assume $-\Delta + V$ has no bound states.

Then, there exists a unique global solution to (NLS) such that

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- Klein-Gordon would work as well. Bound states and radiation damping?

Related works and comments

- Some related works:
  - Soffer-Weinstein ('99), Tsai-Yau ('02) . . . : cubic NLS/KG radiation damping.
  - Cuccagna, Bambusi-Cuccagna ('11): Small Energy Scattering NLS/KG
  - Gustafson-Nakanishi-Tsai ('04): cubic NLS stability of small solitary waves
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- In our result
  - V is large and cannot be treated perturbatively;
  - For u^2, even with $V = 0$, there are fully resonant interactions (at the origin).

Multilinear structure in 3d

- For $x \in \mathbb{R}^3$, generalized eigenfunctions solve

$$\psi(x,k) = e^{ix\cdot k} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} V(y)\psi(y,k)\,\mathrm{d}y$$
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- Leading order contribution

$$\mu(k,\ell,m) = \int_{\mathbb{R}^3} \overline{\psi(x,k)}\psi(x,\ell)\psi(x,m)\,\mathrm{d}x$$
$$\approx \delta(k-\ell-m) + \int_{\mathbb{R}^3} \frac{1}{|x|} e^{i|m||x|} e^{ix\cdot(-k+\ell)} g_0(\omega,m)\,\mathrm{d}x$$

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Thank You for your attention!