Traveling wave solutions to the free boundary incompressible Navier-Stokes equations

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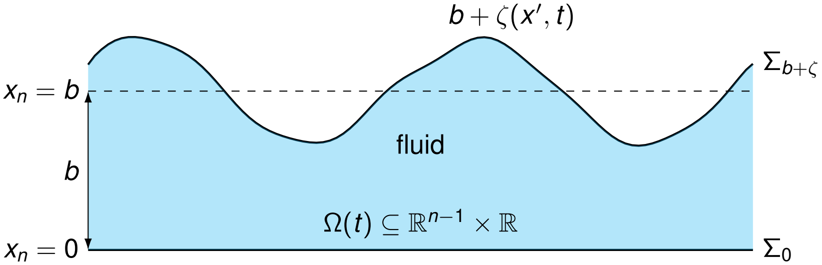
Unknown fluid domain

Consider a layer of incompressible viscous fluid in $\Omega(t) \subset \mathbb{R}^n$, $n \in \{2, 3\}$.



- horizontal cross section is $\mathbb{R}^{n-1}$, equilibrium depth is $b > 0$
- lower surface fixed at $\Sigma_0 = \{x_n = 0\}$
- unknown free surface function $\zeta : \mathbb{R}^{n-1} \times [0, \infty) \to (-b, \infty)$
- upper free surface at $\Sigma_{b+\zeta} = \{x_n = b + \zeta(x', t)\}$
- fluid domain is $\Omega(t) = \{x \in \mathbb{R}^n \mid 0 < x_n < b + \zeta(x', t)\}$

Modeling, unknowns, and stresses

- We assume that the fluid is incompressible and viscous.
- Unknowns for each $t \geq 0$:
  - Free surface function: $\zeta(\cdot, t) : \mathbb{R}^{n-1} \to (-b, \infty)$
  - Fluid velocity: $w(\cdot, t) : \Omega(t) \to \mathbb{R}^n$
  - Fluid pressure $P(\cdot, t) : \Omega(t) \to \mathbb{R}$
- Forces and stresses:
  - Constant gravity: $-\rho \mathfrak{g} e_n$
  - Constant external pressure: $P_{ext} \in \mathbb{R}$
  - External surface stress: $T_{ext}(\cdot, t) : \Sigma_{b+\zeta} \to \mathbb{R}^{n \times n}_{sym}$
  - Surface tension on $\Sigma_{b+\zeta}$:

$$\sigma \mathcal{H}(\zeta) = \sigma \operatorname{div}' \left(\frac{\nabla' \zeta}{\sqrt{1 + |\nabla' \zeta|^2}} \right)$$

    for $\sigma \geq 0$ the coefficient of surface tension
- By rescaling we may assume viscosity $\mu = 1$, density $\rho = 1$, and gravity constant $\mathfrak{g} = 1$.

Free boundary incompressible Navier-Stokes

Equations of motion:

$$\begin{cases} \partial_t w + w \cdot \nabla w - \Delta w + \nabla P = -e_n & \text{in } \Omega(t) \\ \operatorname{div} w = 0 & \text{in } \Omega(t) \\ (PI_{n\times n} - \mathbb{D}w)\nu = -\sigma \mathcal{H}(\zeta)\nu + (P_{ext} I_{n\times n} + T_{ext})\nu & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\ \partial_t \zeta = w \cdot \nu \sqrt{1 + |\nabla' \zeta|^2} & \text{on } \Sigma_{b+\zeta(\cdot,t)} \\ w = 0 & \text{on } \Sigma_0. \end{cases}$$

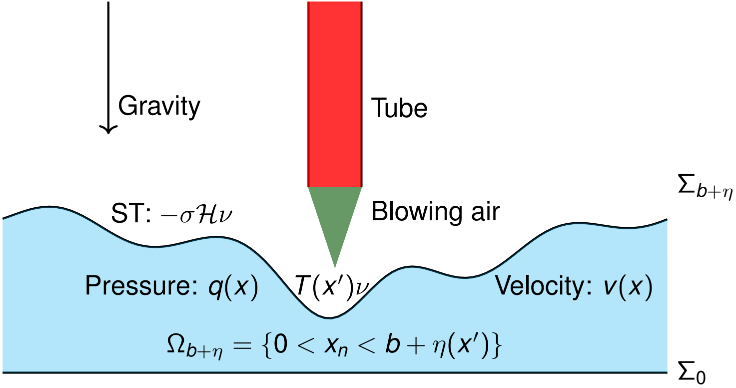
- Here ν is the outward-pointing normal to $\Sigma_{b+\zeta}$.
- Define the viscous stress tensor $S = S(P, w) = PI_{n\times n} - \mathbb{D}w \in \mathbb{R}^{n\times n}_{sym}$, where $\mathbb{D}w = Dw + (Dw)^\intercal$ is the symmetrized gradient. Then

$$\operatorname{div} S(P, w) = \nabla P - \Delta w - \nabla \operatorname{div} w.$$

Traveling wave ansatz

- We assume that the external stress T_{ext} is stationary (time independent) when viewed in a coordinate system moving parallel to Σ_0 with velocity γe_1 for $\gamma \in \mathbb{R}$.
- This models a source of stress translating uniformly above the fluid's surface. We can think of this as:
  - tube of air blowing on the fluid,
  - a very simple model of wind,
  - a ship moving at constant speed,
  - your favorite variant.
- Assume then that $T_{ext}(x,t) = T(x' - \gamma t e_1)$ for $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n\times n}_{sym}$.
- Traveling wave ansatz: $\zeta(x',t) = \eta(x' - \gamma t e_1)$, $w(x,t) = v(x - \gamma t e_1)$, and $P(x,t) = q(x - \gamma t e_1) + P_{ext} - (x_n - b)$ for new unknowns η, v, q.
- Stationary fluid domain: $\Omega_{b+\eta} = \{x \in \mathbb{R}^n \mid 0 < x_n < b + \eta(x')\}$.

Traveling wave cartoon



Stationary problem in the moving coordinate system

Traveling wave equations

The traveling wave equations become the stationary nonlinear elliptic system:

$$(TW)_\gamma : \begin{cases} (v - \gamma e_1) \cdot \nabla v - \Delta v + \nabla q = 0 & \text{in } \Omega_{b+\eta} \\ \operatorname{div} v = 0 & \text{in } \Omega_{b+\eta} \\ (qI_{n\times n} - \mathbb{D} v)\mathcal{N} = (\eta - \sigma \mathcal{H}(\eta))\mathcal{N} + T\mathcal{N} & \text{on } \Sigma_{b+\eta} \\ -\gamma \partial_1 \eta = v \cdot \mathcal{N} & \text{on } \Sigma_{b+\eta} \\ v = 0 & \text{on } \Sigma_0, \end{cases}$$

where here we have written the non-unit normal to $\Sigma_{b+\eta}$ as

$$\mathcal{N} = (-\nabla' \eta, 1) \in \mathbb{R}^n.$$

- Trivial solutions: $\gamma \in \mathbb{R}$, $T = 0$, $v = 0$, $q = 0$, $\eta = 0$
- In fact, in standard Sobolev spaces, if $T = 0$ then $v = 0$, $q = 0$, $\eta = 0$, so an external source of stress / force is needed to overcome the dissipative effects of viscosity.

A woefully brief history

For the inviscid ($\mu = 0$) problem, MUCH is known.

- 2D, irrotational: Nekrasov, Levi–Civita, Krasovskiĭ, Keady-Norbury, Toland, Amick-Toland, Amick-Fraenkel-Toland, Plotnikov, McLeod, Beale
- 2D, rotational: Constantin-Strauss, Wahlén, Walsh, Hur, Groves-Wahlén, Wheeler, Chen-Walsh-Wheeler
- 2D, surface forcing / wind: Wheeler, Walsh-Bühler-Shatah-Zeng
- 3D, irrotational: Ioss-Plotnikov, Groves-Sun, Buffoni-Groves-Sun-Wahlén

Much less is known for the viscous problem.

- $\gamma = 0$ (stationary solutions with forcing): Jean, Pileckas, Nazarov-Pileckas, Pileckas-Zaleskis, Gellrich
- $\gamma \neq 0$ (experimental evidence with translating tube): Park-Cho, Masnadi-Duncan, Akylas-Cho-Dioro-Duncan

To the best of our knowledge, there were no mathematical results for the viscous traveling ($\gamma \neq 0$) problem.

Theorem statement, special case

Theorem (Leoni-T., '19)

Suppose that either $n = 2$ and $\sigma \geq 0$ or $n = 3$ and $\sigma > 0$. Let $n/2 < s \in \mathbb{N}$. Then there exists an open set

$$W^s \subset (\mathbb{R}\backslash\{0\}) \times H^{s+1/2}(\mathbb{R}^{n-1}; \mathbb{R}^{n\times n}_{sym})$$

and a Sobolev-type Banach space $\mathcal{X}^s$ such that the following hold.

- *We have that $(\mathbb{R}\backslash\{0\}) \times \{0\} \subset W^s$, i.e. trivial solutions are in W^s.*
- *$\mathcal{X}^s$ supercritical embedding: $(v, q, \eta) \in \mathcal{X}^s \Rightarrow$*

$$v \in C_b^{2+\lfloor s-n/2 \rfloor}, \; q \in C_b^{1+\lfloor s-n/2 \rfloor}, \; \eta \in C_0^{3+\lfloor s-n/2 \rfloor}.$$

- *For each $(\gamma, T) \in W^s$ there exists a traveling wave solution $(v, q, \eta) \in \mathcal{X}^s$ solving the equations $(TW)_\gamma$ classically. These solutions are locally unique and satisfy various estimates, e.g. $\max|\eta| \leq b/2$.*
- *The map $W^s \ni (\gamma, T) \mapsto (v, q, \eta) \in \mathcal{X}^s$ is locally Lipschitz.*

Theorem statement, remarks

- $\eta(x') \to 0$ as $|x'| \to 0$, so these are "solitary waves."
- The space $\mathcal{X}^s$ involves new Sobolev-type spaces with strange properties (more later).
- Our technique does not work for $\gamma = 0$. This means we can only produce traveling wave solutions and not stationary solutions. The parameter $\gamma \neq 0$ plays an essential role in defining the topology of $\mathcal{X}^s$.
- In the paper we actually prove a more general result with a more general form of the stress and bulk forces as well.
- The results remain true for any $n \geq 2$ when $\sigma > 0$.
- Recent work with N. Stevenson extends this to multi-layer fluids.

Flattening

The first step in proving the theorem is to rephrase $(TW)_\gamma$ in the fixed (equilibrium) domain

$$\Omega := \Omega_b = \{x \in \mathbb{R}^n \mid 0 < x_n < b\} = \mathbb{R}^{n-1} \times (0, b).$$

We do so with the flattening map $\mathfrak{F} : \bar{\Omega} \to \bar{\Omega}_{b+\eta}$ given by

$$\mathfrak{F}(x) = (x', x_n(1 + \eta(x')/b)) = x + \frac{x_n \eta(x')}{b} e_n.$$

Note

$$\mathfrak{F}(\Sigma_0) = \Sigma_0 \text{ and } \mathfrak{F}(\Sigma_b) = \Sigma_{b+\eta}.$$

Also, $\mathfrak{F}$ inherits the regularity of η and is a bijection if $\inf \eta > -b$. We then change unknowns again: $u : \Omega \to \mathbb{R}^n$ and $p : \Omega \to \mathbb{R}$ via

$$u = v \circ \mathfrak{F} \text{ and } p = q \circ \mathfrak{F}.$$

Flattening

We arrive at the equivalent flattened traveling wave system:

$$(FTW)_\gamma : \begin{cases} (u - \gamma e_1) \cdot \nabla_{\mathcal{A}} u + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(p,u)\mathcal{N} = (\eta - \sigma \mathcal{H}(\eta))\mathcal{N} + T\mathcal{N} & \text{on } \Sigma_b \\ u \cdot \mathcal{N} + \gamma \partial_1 \eta = 0 & \text{on } \Sigma_b \\ u = 0 & \text{on } \Sigma_0. \end{cases}$$

Here

- $\mathcal{A} = (D\mathfrak{F})^{-\intercal}$ and $\partial_i \mapsto \mathcal{A}_{ij}\partial_j$, which defines $\nabla_{\mathcal{A}}$, $\operatorname{div}_{\mathcal{A}}$, $\Delta_{\mathcal{A}}$, etc.
- $S_{\mathcal{A}}(p,u) = pI - \mathbb{D}_{\mathcal{A}} u$ and $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u - \nabla_{\mathcal{A}} \operatorname{div}_{\mathcal{A}} u$.
- In this form we see the problem is a quasilinear elliptic system.

Linearization

The strategy is to use the implicit function theorem to solve for (u, p, η) in terms of (γ, T). As a first step we linearize in (u, p, η) around the trivial solution $\gamma \in \mathbb{R}$, $T = 0$, $u = 0$, $p = 0$, $\eta = 0$ to get $\gamma-$Stokes with traveling gravity-capillary BCs:

$$(TGC)_\gamma : \begin{cases} \operatorname{div} S(p,u) - \gamma \partial_1 u = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \\ u_n + \gamma \partial_1 \eta = h & \text{on } \Sigma_b \\ S(p,u)e_n - (\eta - \sigma \Delta' \eta) e_n = k & \text{on } \Sigma_b \\ u = 0 & \text{on } \Sigma_0. \end{cases}$$

Here (f, g, h, k) are data for the linearized problem.

A faulty start

At first glance it looks like we should set $\gamma = 0$ and decouple:

$$\begin{cases} \operatorname{div} S(p,u) = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \\ u_n = h, \text{ and } \; -(\mathbb{D}ue_n)' = (S(p,u)e_n)' = k' & \text{on } \Sigma_b \\ u = 0 & \text{on } \Sigma_0 \end{cases}$$

$$\text{and} \qquad \eta - \sigma\Delta'\eta = p - \mathbb{D}ue_n \cdot e_n - k_n \text{ on } \Sigma_b.$$

This runs into a fatal problem.

- Lack of p BC in first system means we only get elliptic estimates $u \in H^{s+2}$, $\nabla p \in H^s$ provided that $f \in H^s$, etc.
- In second equation we have the trace of p onto Σ_b. What is the trace space for the homogeneous Sobolev space $\dot{H}^1(\Omega)$?
- In Leoni-T. (*JFA* '19) we exactly characterize this trace space. It is a nonstandard fractional homogeneous Sobolev-type space. If we use it to solve for η above we can't guarantee η is bounded, or even a function!

Overdetermined problem, pt. 1

If we return to $\gamma \neq 0$, then the decoupling isn't possible. To understand what's happening in the full linear problem we initially ignore η, which leads us to consider the overdetermined problem

$$(ODP)_\gamma : \begin{cases} \operatorname{div} S(p,u) - \gamma \partial_1 u = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \\ S(p,u) e_n = k, \quad u_n = h & \text{on } \Sigma_b \\ u = 0 & \text{on } \Sigma_0. \end{cases}$$

Why is this overdetermined?

- We specify $n+1$ boundary conditions on Σ_b but n on Σ_0.
- If we ignore the equation $u_n = h$ on Σ_b, then we have the $\gamma-$Stokes system with stress BCs, and this problem is well-posed...

$\gamma-$Stokes with stress BCs

Theorem ($\gamma-$Stokes is well-posed)

For every $\gamma \in \mathbb{R}$ and every $s \geq 0$, the bounded linear operator

$$\Phi_\gamma : {}_0H^{s+2}(\Omega; \mathbb{R}^n) \times H^{s+1}(\Omega) \to H^s(\Omega; \mathbb{R}^n) \times H^{s+1}(\Omega) \times H^{s+1/2}(\Sigma_b; \mathbb{R}^n)$$

given by

$$\Phi_\gamma(u, p) = (\operatorname{div} S(p, u) - \gamma \partial_1 u, \operatorname{div} u, S(p, u) e_n|_{\Sigma_b})$$

is an isomorphism.

Here

$${}_0H^{s+2}(\Omega; \mathbb{R}^n) = \{u \in H^{s+2}(\Omega; \mathbb{R}^n) \mid u = 0 \text{ on } \Sigma_0\},$$

so the no-slip BC is enforced automatically.

Overdetermined problem, pt. 2

Consequently, we cannot solve the overdetermined problem in general. There will be compatibility conditions on the data! To see the first consider $\operatorname{div} u = g$ and $u_n = h$ on Σ_b and $u_n = 0$ on Σ_0.

- If all terms were L^1 we could compute

$$\int_\Omega g = \int_\Omega \operatorname{div} u = \int_{\Sigma_b} u_n = \int_{\Sigma_b} h \Leftrightarrow \int_{\mathbb{R}^{n-1}} \left(h - \int_0^b g(\cdot, x_n) dx_n \right) dx' = 0$$

  However, we work in L^2-based spaces in Ω, which has infinite measure, so these aren't valid computations.
- Playing games with test functions, we can derive the appropriate L^2 formulation of the first compatibility condition:

$$h - \int_0^b g(\cdot, x_n) dx_n \in \dot{H}^{-1}(\mathbb{R}^{n-1}),$$

  where $\dot{H}^{-1}(\mathbb{R}^{n-1})$ is the homogeneous Sobolev space of order -1. This is a weak form of the above condition defined via the Fourier transform:

$$\hat{\varphi}(0) = 0 \Leftrightarrow \int_{\mathbb{R}^{n-1}} \varphi(x') dx' = 0.$$

Overdetermined problem, pt. 3

This still isn't enough. We now take a cue from the closed range theorem. The formal adjoint of the overdetermined problem is the underdetermined problem (here in homogeneous form):

$$(UDP)_\gamma : \begin{cases} \operatorname{div} S(q,v) + \gamma \partial_1 v = 0 & \text{in } \Omega \\ \operatorname{div} v = 0 & \text{in } \Omega \\ (S(q,v)e_n)' = 0 & \text{on } \Sigma_b \\ v = 0 & \text{on } \Sigma_0. \end{cases}$$

- Underdetermined because we specify only $n-1$ BCs on Σ_b.
- We exactly characterize the space of solutions by adding the condition $S(q,v)e_n \cdot e_n = \varphi$. Then $(v,q) = \Phi_\gamma^{-1}(0,0,\varphi e_n)$.
- This leads to a second compatibility condition as in closed range theorem (mult, IBP): if there's a solution (u,p) to $(ODP)_\gamma$ with data (f,g,h,k), then

$$\int_\Omega (f \cdot v - gq) - \int_{\Sigma_b} (k \cdot v - h\varphi) = 0$$

for all $\varphi \in H^{s+1/2}$ and $(v,q) = \Phi_\gamma^{-1}(0,0,\varphi e_n)$.

Overdetermined problem, pt. 4

These two CCs are necessary and sufficient!

Theorem ($(ODP)_\gamma$ is well-posed with CCs)

Let $\gamma \in \mathbb{R}$ and $s \geq 0$. The problem $(ODP)_\gamma$ is uniquely solvable for $u \in {}_0H^{s+2}(\Omega; \mathbb{R}^n)$ and $p \in H^{s+1}(\Omega)$ if and only if

$$(f, g, h, k) \in H^s(\Omega; \mathbb{R}^n) \times H^{s+1}(\Omega) \times H^{s+3/2}(\Sigma_b) \times H^{s+1/2}(\Sigma_b; \mathbb{R}^n)$$

satisfy the two CCs:

$$h - \int_0^b g(\cdot, x_n) dx_n \in \dot{H}^{-1}(\mathbb{R}^{n-1}) \text{ and } \int_\Omega (f \cdot v - gq) - \int_{\Sigma_b} (k \cdot v - h\varphi) = 0 \text{ for all } \varphi.$$

Moreover, we get an isomorphism

$${}_0H^{s+2}(\Omega; \mathbb{R}^n) \times H^{s+1}(\Omega) \ni (u, p) \mapsto (f, g, h, k) \in \mathcal{Z}^s,$$

where $\mathcal{Z}^s$ encodes the regularity and both CCs.

Fourier version

We now want to return η to the overdetermined problem, but the second *CC* is hard to work with. Reformulate on Fourier side: second *CC* is equivalent to

$$\int_0^b (\hat{f}(\xi,x_n)\cdot\overline{V(\xi,x_n,-\gamma)}-\hat{g}(\xi,x_n)\overline{Q(\xi,x_n,-\gamma)})dx_n-\hat{k}(\xi)\cdot\overline{V(\xi,b,-\gamma)}+\hat{h}(\xi)=0$$

for almost every $\xi \in \mathbb{R}^{n-1}$.

- $\hat{\cdot}$ denotes the horizontal Fourier transform.
- Q and V are special solutions to an ODE (with variable $x_n \in (0,b)$) corresponding to the FT of the $\gamma-$Stokes problem with data $f=0$, $g=0$, $k'=0$, and $\hat{k}_n=1$.
- φ is now gone, so we can easily work with the condition.

Return to $\gamma-$Stokes with traveling grav.-cap. BCs

If a solution (u, p, η) exists for

$$(TGC)_\gamma : \begin{cases} \operatorname{div} S(p,u) - \gamma \partial_1 u = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \\ u_n + \gamma \partial_1 \eta = h & \text{on } \Sigma_b \\ S(p,u)e_n - (\eta - \sigma \Delta' \eta) e_n = k & \text{on } \Sigma_b \\ u = 0 & \text{on } \Sigma_0, \end{cases}$$

then the CCs must hold for

$$f, g, h - \gamma \partial_1 \eta, \text{ and } k + (\eta - \sigma \Delta' \eta) e_n.$$

The first CC is no problem because $\gamma \partial_1 \eta \in \dot{H}^{-1}(\mathbb{R}^{n-1})$. The second is trickier...

Return to $\gamma-$Stokes with traveling grav.-cap. BCs

The second CC holds if and only if

$$\rho(\xi)\hat{\eta}(\xi) = \psi(\xi) \text{ for } \xi \in \mathbb{R}^{n-1},$$

where $\psi, \rho : \mathbb{R}^{n-1} \to \mathbb{C}$ are given by

$$\psi(\xi) = \int_0^b \left(\hat{f}(\xi, x_n) \cdot \overline{V(\xi, x_n, -\gamma)} - \hat{g}(\xi, x_n)\overline{Q(\xi, x_n, -\gamma)}\right) dx_n \\ - \hat{k}(\xi) \cdot \overline{V(\xi, b, -\gamma)} + \hat{h}(\xi),$$

and

$$\rho(\xi) = 2\pi i \gamma \xi_1 + (1 + 4\pi^2 \sigma |\xi|^2)\overline{V_n(\xi, b, -\gamma)}.$$

Return to $\gamma-$Stokes with traveling grav.-cap. BCs

The function $V_n(\cdot, b, \gamma)$ is the symbol associated to the pseudodifferential operator

$$H^s(\Sigma_b) \ni \varphi \mapsto u_n|_{\Sigma_b} \in H^{s+1}(\Sigma_b),$$

where $(u, p) \in H^{s+3/2}(\Omega; \mathbb{R}^n) \times H^{s+1/2}(\Omega)$ solve

$$\begin{cases} \operatorname{div} S(p,u) - \gamma \partial_1 u = 0 & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ S(p,u) e_n = \varphi e_n & \text{on } \Sigma_b \\ u = 0 & \text{on } \Sigma_0, \end{cases}$$

which is the normal-stress to normal-Dirichlet map. Thus the pseudodifferential operator $\rho(\nabla)$ synthesizes:

- traveling wave boundary operator $\gamma \partial_1$ ($\rightsquigarrow 2\pi i \gamma \xi_1$ in ρ),
- gravity-capillary operator $I - \sigma \Delta'$ ($\rightsquigarrow 1 + 4\pi^2 \sigma |\xi|^2$ in ρ),
- normal-stress to normal-Dirichlet operator ($\rightsquigarrow \overline{V_n(\xi, b, -\gamma)}$ in ρ).

ρ asymptotics

To solve $\rho\hat{\eta} = \psi$ we need to understand the behavior of ρ. A very lengthy and painful set of calculations reveals:

- $\rho(\xi) = 0$ if and only if $\xi = 0$.
- If $\sigma > 0$ and $n \geq 3$ then

$$\left|\rho(\xi)\right|^2 \asymp \begin{cases} \gamma^2 \xi_1^2 + \left|\xi\right|^4 & \text{for } \left|\xi\right| \asymp 0 \\ 1 + \sigma^2 \left|\xi\right|^2 & \text{for } \left|\xi\right| \asymp \infty. \end{cases}$$

- If $\sigma \geq 0$ and $n = 2$,

$$\left|\rho(\xi)\right|^2 \asymp \begin{cases} \gamma^2 \left|\xi\right|^2 + \left|\xi\right|^4 & \text{for } \left|\xi\right| \asymp 0 \\ 1 + [\gamma^2 + \sigma^2] \left|\xi\right|^2 & \text{for } \left|\xi\right| \asymp \infty. \end{cases}$$

Key point: in $\mathbb{R} = \mathbb{R}^{2-1}$ the operator $\gamma\partial_1$ is elliptic.

η estimates

For $\sigma > 0$, $n \geq 3$ we get

$$\int_{B(0,1)} \frac{\gamma^2 \xi_1^2 + |\xi|^4}{|\xi|^2} |\hat{\eta}(\xi)|^2 \, d\xi + \int_{B(0,1)^c} (1 + \sigma^2 |\xi|^2)^{s+5/2} |\hat{\eta}(\xi)|^2 \, d\xi$$
$$\asymp \int_{B(0,1)} \frac{1}{|\xi|^2} |\psi(\xi)|^2 \, d\xi + \int_{B(0,1)^c} (1 + |\xi|^2)^{s+3/2} |\psi(\xi)|^2 \, d\xi,$$

while for $\sigma \geq 0$ and $n = 2$ we get

$$\int_{B(0,1)} (\gamma^2 + |\xi|^2) |\hat{\eta}(\xi)|^2 \, d\xi + \int_{B(0,1)^c} (1 + \gamma^2 |\xi|^2)^{s+5/2} |\hat{\eta}(\xi)|^2 \, d\xi$$
$$\asymp \int_{B(0,1)} \frac{1}{|\xi|^2} |\psi(\xi)|^2 \, d\xi + \int_{B(0,1)^c} (1 + |\xi|^2)^{s+3/2} |\psi(\xi)|^2 \, d\xi.$$

Fortunately, we can control RHS of each using regularity and first CC. Moral: $n = 2$ gives standard $H^{s+5/2}$ estimate, $n = 3$ gives something weird. Here $\gamma \neq 0$ is essential.

Specialized anisotropic Sobolev space

For $s \geq 0$ we define $X^s(\mathbb{R}^d)$ via the norm

$$\|f\|_{X^s}^2 = \int_{B(0,1)} \frac{\xi_1^2 + |\xi|^4}{|\xi|^2} \left|\hat{f}(\xi)\right|^2 d\xi + \int_{B(0,1)^c} (1 + |\xi|^2)^s \left|\hat{f}(\xi)\right|^2 d\xi.$$

We then prove:

- This is a Hilbert space and $H^s(\mathbb{R}^d) \subset X^s(\mathbb{R}^d)$ for $d \geq 2$.
- Technical miracle:
$$\int_{B(0,1)} \frac{|\xi|^2}{\xi_1^2 + |\xi|^4} d\xi < \infty.$$
Thus, elements of $X^s(\mathbb{R}^d)$ are actual functions! In fact, $X^s(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d) + C_0^\infty(\mathbb{R}^d)$ (goes to 0, not compact support).
- It's not closed under composition with rotation.
- Good embedding properties. E.g. $s > k + d/2 \Rightarrow H^s(\mathbb{R}^d) \hookrightarrow C_0^k(\mathbb{R}^d)$, etc.
- Good nonlinear functional analytic properties (products, etc).

Specialized spaces in Ω

For $s \geq 0$ we also define the Banach space

$$Y^s(\Omega) = H^s(\Omega) + X^s(\mathbb{R}^{n-1}) = \{q(x', x_n) + \zeta(x') \mid q \in H^s(\Omega), \zeta \in X^s(\mathbb{R}^{n-1})\}.$$

- These spaces inherit all of the properties of $X^s(\mathbb{R}^{n-1})$, so they aren't that bad.
- We need these because of the appearance of both p and η in the stress BC. We end up getting

$$p - \eta \in H^{s+1}(\Omega) \Rightarrow p \in Y^{s+1}(\Omega).$$

- This means that both the free surface and the pressure are in unusual spaces, but for p we know exactly how / why, and $p - \eta$ is in a normal Sobolev space.

Isomorphism for the main linear problem

Theorem ($(TGC)_\gamma$ is well-posed if $\gamma \neq 0$)

Assume $\gamma \neq 0$. For $s \geq 0$ we define the Banach subspace

$$\mathcal{X}^s \subset {}_0H^{s+2}(\Omega;\mathbb{R}^n) \times Y^{s+1}(\Omega) \times X^{s+5/2}(\mathbb{R}^{n-1})$$

in which $p - \eta \in H^{s+1}(\Omega)$. There exists a Banach space $\mathcal{Y}^s$, encoding the regularity and the first CC, such that the map $\Upsilon_{\gamma,\sigma} : \mathcal{X}^s \to \mathcal{Y}^s$ given by

$$\Upsilon_{\gamma,\sigma}(u,p,\eta) \\ = (\operatorname{div} S(p,u) - \gamma\partial_1 u, \operatorname{div} u, \, u_n|_{\Sigma_b} + \gamma\partial_1\eta, \, S(p,u)e_n|_{\Sigma_b} - (\eta - \sigma\Delta'\eta)e_n).$$

is an isomorphism when either $\sigma > 0$ and $n \geq 2$ or $\sigma = 0$ and $n = 2$.

Note: when $n = 2$, $X^{s+5/2}(\mathbb{R}) = H^{s+5/2}(\mathbb{R})$ and $Y^{s+1}(\Omega) = H^{s+1}(\Omega)$.

Nonlinear analysis

Recall that, given $\gamma \neq 0$ and external stress T, we want to solve

$$(FTW)_\gamma : \begin{cases} (u - \gamma e_1) \cdot \nabla_{\mathcal{A}} u + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(p, u)\mathcal{N} - (\eta - \sigma \mathcal{H}(\eta))\mathcal{N} - T\mathcal{N} = 0 & \text{on } \Sigma_b \\ u \cdot \mathcal{N} + \gamma \partial_1 \eta = 0 & \text{on } \Sigma_b \\ u = 0 & \text{on } \Sigma_0. \end{cases}$$

We formulate this as an implicit function argument,

$$\Xi(\gamma, T, u, p, \eta) = (0, 0, 0, 0)$$

for

$$\Xi : [\mathbb{R} \times H^{s+1/2}(\Sigma_b; \mathbb{R}^{n \times n}_{\text{sym}})] \times U^s \to \mathcal{Y}^s$$

for $U^s \subset \mathcal{X}^s$ an open set with η sufficiently small that (among other things) $\|\eta\|_{C^3_0} \leq b/2$, which means the flattening map is a C^3 diffeomorphism.

Nonlinear analysis

The full map is given by

$$\Xi(\gamma, T, u, p, \eta) = ((u - \gamma e_1) \cdot \nabla_{\mathcal{A}} u + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u), J \operatorname{div}_{\mathcal{A}} u,$$
$$u \cdot \mathcal{N} + \gamma \partial_1 \eta, (pI - \mathbb{D}_{\mathcal{A}} u)\mathcal{N} - (\eta - \sigma \mathcal{H}(\eta))\mathcal{N} - T\mathcal{N}),$$

where $\mathcal{A}$, $\mathcal{N}$, $J = \det D\mathfrak{F}$ etc are determined by η.

- Functional analytic properties of $X^{s+5/2}(\mathbb{R}^{n-1})$ and $Y^{s+1}(\Omega)$ are essential for Ξ to be well-defined and C^1.
- Enforcing the first linearized CC in the nonlinear problem is a technical trick: $J \operatorname{div}_{\mathcal{A}} u$ and $u \cdot \mathcal{N}$ enjoy a nonlinear analog of $\operatorname{div} u$ and $u \cdot e_n$.
- Implicit function theorem essentials: $\Xi(\gamma, 0, 0, 0, 0) = (0, 0, 0, 0)$ and

  $$D_2\Xi(\gamma, 0, 0, 0, 0)(\dot{u}, \dot{p}, \dot{\eta}) = \Upsilon_{\gamma,\sigma}(\dot{u}, \dot{p}, \dot{\eta}),$$

  which is an isomorphism.
- Finally, map back to the original, non-flattened problem with the help of smallness of η.

Thanks!

Thank you for your attention!