Global mild solutions of the Landau and non-cutoff Boltzmann equation

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The Boltzmann equation (1872)

- The Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad x \in \Omega,\ v \in \mathbb{R}^3,\ t \geq 0$$

- $F = F(t, x, v)$: probability density in (position,velocity)
- $\Omega \subset \mathbb{R}^3$: domain in space
- $v \cdot \nabla_x F$: free transport term
- $Q(F, F)$: collision operator, local in (t, x), quadratic integral operator
- Derived from **rarefied gas dynamics**: *Maxwell 1860'*, *Boltzmann 1872*.

Physical conservation laws of energy and momentum

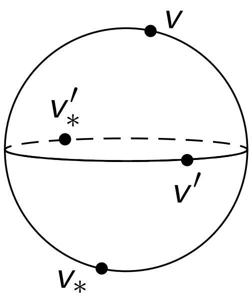
Given two particles, with velocities $v, v_* \in \mathbb{R}^3$, after a binary collision they have outgoing velocities $v', v'_* \in \mathbb{R}^3$.

- These obey the conservation of momentum and energy:

$$\begin{aligned} v + v_* &= v' + v'_*, \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2. \end{aligned}$$

- Since there are six unknowns, (v', v'_*), and four equations, the set of solutions can be parametrized on the sphere $\sigma \in \mathbb{S}^2$:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma,$$
$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$



- Describes the aftermath of a binary collision probabilistically.

Boltzmann collision kernel: $B(v - v_*, \sigma)$

$$\mathcal{Q}(F,G)(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma \, B(v - v_*, \sigma) \, [F'_* G' - F_* G] \, .$$

J. C. Maxwell in 1866 computed $B(v - v_*, \sigma)$ from the potential:

$$\phi(r) = r^{-(p-1)}, \quad p \in (2, +\infty).$$

This kernel takes product form in its arguments as

$$B(v - v_*, \sigma) = |v - v_*|^\gamma \, b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

The angular singularity in σ is not locally integrable:

$$\sin\theta b(\cos\theta) \approx \theta^{-1-2s}, \quad s = \frac{1}{p-1} \in (0,1), \quad \forall\, \theta \in \left(0, \frac{\pi}{2}\right] .$$

- The kinetic factor $|v - v_*|^\gamma$ can be singular: $\gamma = \frac{p-5}{p-1} > -3$.
- **In our results we assume $\gamma + 2s > -\frac{3}{2}$.**
- **Grad angular cutoff assumption**

Landau collision operator

$\boxed{\phi(r) = r^{-(p-1)}\ (p>2):}$ $p \to 2^+$ corresponds to the **Coulomb potential** when $\gamma \to -3$ and $s \to 1$. In the limit the Boltzmann operator is no longer valid, replaced by the **Landau collision operator (1936)**:

$$Q(g,h) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \psi(v-u) \left[g(u) \nabla_v h(v) - h(v) \nabla_u g(u) \right] du \right\}$$

$$= \sum_{j,m=1}^{3} \partial_{v_j} \int_{\mathbb{R}^3} \psi^{jm}(v-u) \left[g(u) \partial_{v_m} h(v) - h(v) \partial_{u_m} g(u) \right] du.$$

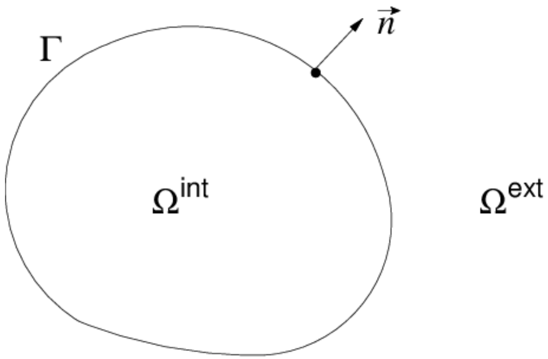
The non-negative matrix ψ is

$$\psi^{jm}(v) = \frac{1}{8\pi} \left(\delta_{jm} - \frac{v_j v_m}{|v|^2} \right) |v|^{\gamma+2}, \quad \gamma \geq -3.$$

$\gamma = -3$ is the physical Coulomb potential case.

Typical domains Ω in space

- The Torus $x \in \mathbb{T}^3 = [0, 2\pi]^3$ with periodic boundary conditions: $F(t, x, v) = F(t, x + 2\pi e_i, v)$ for $i = 1, 2, 3$.
- The whole space $x \in \mathbb{R}^3$ with suitable decay at infinity $F(t, x, v) \to 0$ as $|x| \to \infty$
- A domain $\Omega \subset \mathbb{R}^3$ with general kinetic boundary conditions as given in the next slides.



General boundary conditions for Kinetic equations

- The spatial domain $\Omega = \{x : \zeta(x) < 0\}$ is connected and bounded with $\zeta(x)$ being a smooth function. (We assume that $\nabla\zeta(x) \neq 0$ at the boundary $\zeta(x) = 0$.)
- Define the outward normal vector $n(x)$ on the boundary $\Gamma = \partial\Omega$ as

$$n(x) \stackrel{\text{def}}{=} \frac{\nabla\zeta(x)}{|\nabla\zeta(x)|}, \quad x \in \partial\Omega.$$

- The phase boundary of $\Omega \times \mathbb{R}^3$ is $\gamma \stackrel{\text{def}}{=} \partial\Omega \times \mathbb{R}^3$.
- Split the boundary into the outgoing boundary γ_+, incoming boundary γ_-, and singular boundary γ_0 for grazing velocities:

outgoing boundary: $\gamma_+ \stackrel{\text{def}}{=} \{(x, v) \in \Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}$,

incoming boundary: $\gamma_- \stackrel{\text{def}}{=} \{(x, v) \in \Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}$,

singular boundary: $\gamma_0 \stackrel{\text{def}}{=} \{(x, v) \in \Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}$.

General boundary conditions for Kinetic equations

For either the Boltzmann or Landau equations (**a.k.a. the Kinetic equations**) we have the four general physical kinetic boundary conditions:

- **In-flow**: $F(t,x,v)|_{\gamma_-} = G(t,x,v)$.
- **Bounce-back**: $F(t,x,v)|_{\gamma_-} = F(t,x,-v)$. (non-physical)
- **Specular reflection**:

$$F(t,x,v)|_{\gamma_-} = F(t,x,R_x v), \;\; R_x v = v - 2(v\cdot n(x))n(x).$$

- **Diffusive reflection**:

$$F(t,x,v)|_{\gamma_-} = \mu(v) \int_{n(x)\cdot v' > 0} F(t,x,v')(n(x)\cdot v')dv'.$$

A large difficulty to study the initial boundary value problems for the Boltzmann equation is that there can be a singularity at the boundary (Guo-Kim-Tonon-Trescaes, 2017), and this singularity may propagate to the interior of the domains (Kim, 2011).

Perturbation framework

We consider the normalized global Maxwellian equilibrium:

$$\mu(v) = (2\pi)^{-3/2} \exp(-|v|^2/2)$$

We make a reformulation of $F(t,x,v)$ as

$$F(t,x,v) = \mu + g(t,x,v), \quad g(t,x,v) = \mu^{1/2} f(t,x,v)$$

Then the perturbation $f = f(t,x,v)$ evolves via the equation

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f,f)$$

where the non-linear and linearized collision operators are

$$\Gamma(f,g) = \mu^{-\frac{1}{2}} \mathcal{Q}(\sqrt{\mu} f, \sqrt{\mu} g), \quad Lf = -\Gamma(f,\sqrt{\mu}) - \Gamma(\sqrt{\mu}, f)$$

Brief review of literature in perturbative framework

- Anglar cutoff case with $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$
  - $0 \leq \gamma \leq 1$: Ukai (1974), Space $X = L^\infty_\beta(H^s_x)$, $s > 3/2$.
  - $-1 < \gamma < 0$: Ukai-Asano (1982), Space: $X = L^\infty_\beta(H^s_x)$, $s > 3/2$.

Approach: Look for a fixed point $f(t, x, v)$ in

$$C([0, \infty); X)$$

to the integral equation

$$f(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} \Gamma(f, f)(s)\, ds,$$

with $B = -v \cdot \nabla_x - L$.

Boltzmann equation with angular cutoff soft potentials $-3 < \gamma < 0$, and Landau equation.

- Guo (2002, 2003, ...), Space $X = H^s_{t,x,v}$, $s \geq 4$.
- Time-decay for soft potentials: Strain-Guo (2006,2008), ...

Approach: Energy method – Obtain uniform a priori estimates

$$\frac{d}{dt}\|f(t)\|_X^2 + \|f(t)\|_{D,X}^2 \lesssim \|f(t)\|_X \|f(t)\|_{D,X}^2.$$

- Trilinear estimates: $\left|(\partial\Gamma(f,f), \partial f)_{L^2_{x,v}}\right| \lesssim \|f(t)\|_X \|f(t)\|_{D,X}^2$,
- Separately estimate the dissipation of the macroscopic part:

$$\mathbf{P}f = \{a_f + b_f \cdot v + c_f |v|^2\}\mu^{1/2},$$

in terms of moment equations:

$$\partial_t a + \nabla \cdot b = \ldots$$
$$\partial_t b + \nabla(a + 2c) + \ldots = \ldots$$
$$\partial_t c + \nabla \cdot b + \ldots = \ldots$$
$$\ldots$$

Non-cutoff Boltzmann equation with $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$

- AMUXY (Series 2011-2012), Space $X = H^4(\mathbb{R}^3_x \times \mathbb{R}^3_v)$

$$
\begin{aligned}
|||f|||^2 \stackrel{\text{def}}{=} & \int_{\mathbb{R}^6\times\mathcal{S}^2} B(v-v_*,\sigma)\mu_*(f'-f)^2 dv_* dv d\sigma \\
& + \int_{\mathbb{R}^6\times\mathcal{S}^2} B(v-v_*,\sigma) f_*^2 \left(\sqrt{\mu'_*} - \sqrt{\mu_*}\right)^2 dv_* dv d\sigma.
\end{aligned}
$$

- Gressman-S. (2011), Space $X = H^2_x L^2_v(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ for the hard potentials and soft potentials from an inverse power law, Norm:

$$
\begin{aligned}
|f|^2_{N^{s,\gamma}} \stackrel{\text{def}}{=} & \left| \langle v\rangle^{\frac{\gamma+2s+1}{2}} f(v) \right|^2_{L^2_v} \\
& + \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv' \left(\langle v\rangle \langle v'\rangle\right)^{\frac{\gamma+2s+1}{2}} \frac{(f(v)-f(v'))^2}{d(v,v')^{3+2s}},
\end{aligned}
$$

with $d(v,v') = \sqrt{|v-v'|^2 + \frac{1}{4}(|v|^2 - |v'|^2)^2}$.

- **Remark that** $|||\{\mathbf{I}-\mathbf{P}\}f|||^2 \approx |\{\mathbf{I}-\mathbf{P}\}f|^2_{N^{s,\gamma}} \approx (f, Lf)_{L^2_v}$.

Polynomial or exponential velocity weight: $\Omega = \mathbb{T}^3$

Existence of g with $F = \mu + g$ such that $m(v)g \in L^1_v L^\infty_x$ (Mild Tail)

- Cutoff case:
  - Arkeryd-Esposito-Pulvirenti (C.M.P., 1987)
  - Gualdani-Mishler-Mouhot (2013) (Mém. Soc. Math. Fr., 2017)
- Landau case:
  - Carrapatoso-Mischler (2017), Space $X = H^2_x L^2_v(\mathbb{T}^3 \times \mathbb{R}^3, m)$.
- Non-cutoff case:
  - Herau-Tonon-Tristani (2017, arXiv:1710.01098): *Cauchy theory and exponential stability for inhomogeneous Boltzmann equation for hard potentials without cut-off.* Space $X = H^3_x L^2_v(\mathbb{T}^3 \times \mathbb{R}^3, m)$.
  - He-Jiang (2017, arXiv:1710.00315): *On the global dynamics of the inhomogeneous Boltzmann equations without angular cutoff: Hard potentials and Maxwellian molecules.*

    Space $X = H^{\frac{3}{2}+}_x L^2_v(\mathbb{T}^3 \times \mathbb{R}^3, m)$.
  - Alonso-Morimoto-Sun-Yang (2018, arXiv:1812.05299): *Non-cutoff Boltzmann equation with polynomial decay perturbation.* Space $X = H^2_x L^2_v(\mathbb{T}^3 \times \mathbb{R}^3, m)$.

Existence of $L^\infty_{x,v}$ solutions in bounded domains

Guo (2010): Hard potentials, cutoff case:

$\gamma \geqslant 0$

$$f(t) \in L^\infty(\Omega, w)$$

Approach: Let $f(t) = U(t)f_0$ solves $\{\partial_t + v \cdot \nabla_x + L\}f = 0$, $f|_{t=0} = f_0$.

- L^2 time-decay: $\|f(t)\|_{L^2} \lesssim e^{-\lambda t}\|f_0\|_{L^2}$
- L^∞ time-decay: Let $L = \nu - K$. For a velocity-growth weight, $h = wf$ solves $\{\partial_t + v \cdot \nabla_x + \nu\}h = K_w h$ with $K_w \overset{\text{def}}{=} wK\frac{1}{w}$. Double duhamel Principle (Vidav, 1970) gives

$$\begin{aligned} U(t) &= G(t) + \int_0^t \mathrm{d}s\, G(t-s)K_w U(s) \\ &= G(t) + \int_0^t \mathrm{d}s\, G(t-s)K_w G(s) \\ &\quad + \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\tau\, G(t-s)K_w G(s-\tau)K_w U(\tau). \end{aligned}$$

The key is to estimate L^∞ norm of the 3rd term.

Lots of recent activity on the boundary value problem

Results for the cutoff Boltzmann equation in a plate

- C. Cercignani [1967], Existence and uniqueness in the large for boundary value problems in kinetic theory.
- Esposito-Lebowitz-Marra [1994], Hydrodynamic limit of the stationary Boltzmann equation in a slab.
- Esposito-Lebowitz-Marra [1995], The Navier-Stokes limit of stationary solutions of the nonlinear Boltzmann equation.

Results in a general bounded domain

- C. Cercignani [1968], Existence, uniqueness, and convergence of the solutions of models in kinetic theory.
- Shizuta and Asano (1977).
- S. Mischler [2010], DiPerna-Lions renormalized solution to the Boltzmann equation, VP and VFP for the maxwell boundary condition with non-constant accommodation coefficient.
- S. Mischler [CMP, 2010] DiPerna-Lions renormalized solution for the initial boundary value problem for VPB.
- Y. Guo [2010], Decay and continuity of the Boltzmann equation in bounded domain by an $L^2 \cap L^\infty$ argument (Four basic boundary conditions with angular cutoff).
- C. Kim [2011], Propagation of discontinuity for Boltzmann equaiton in non-convex domain. Kim (2013).
- Esposito-Guo-Kim-Marra [2013], Stationary solution of the Boltzmann equation with diffuse reflection boundary condition.
- Briant (2014), exponential lower bound (non-cutoff)
- Guo-Kim-Tonon-Trescases [2017], Regularity of the Boltzmann equation in convex domain.
- Briant-Guo [2017], Maxwell boundary condition for the Boltzmann equation ($\alpha \in (\sqrt{2/3}, 1]$).
- Jiang-Zhang (2017), Global existence of renormalized solution (non-cutoff)
- Liu-Yang [2017], The initial boundary value problem for the Boltzmann equation with soft potential.
- Duan-Wang [2018], The Boltzmann equation with large-amplitude initial data in bounded domains.
- Esposito-Guo-Kim-Marra [2018], Stationary solutions to the Boltzmann equation in the Hydrodynamic limit.
- Kim and Lee (2018, 2019) Boltzmann equation with specular boundary condition in convex domains
- Guo-Hwang-Jang-Ouyang [2020, ARMA], On the Landau equation with the specular reflection boundary condition
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Besov space existence theories for Boltzmann

Low regularity: (replace $X = H_x^{3/2+}$ with X still embedded into L_x^∞):

- Duan-Liu-Xu (ARMA 2016): GWP in critical Besov space ($X = B_{2,1}^{3/2}$). for cutoff Boltzmann equation with hard potentials
- Morimoto-Sakamoto (JDE 2016): non-cutoff, hard potentials
- Duan-Sakamoto (KRM 2018): non-cutoff, soft potentials

Approach: energy-spectrum (Besov spaces):

$$F = \mu + \mu^{1/2} f,$$

$$\|f\|_T \stackrel{\text{def}}{=} \sup_{q \geq -1} 2^{qs} \sup_{0 \leq t \leq T} \|\Delta_q f(t, \cdot, \cdot)\|_{L^2_{x,v}}, \quad s = \frac{3}{2},$$

motivated by the Chemin-Lerner space for treating the INS.

$$\sum_{q \geq -1} 2^{qs} \sup_{0 \leq t \leq T} \|\Delta_q u(t)\|_{L^2_x} < \infty.$$

New space: Wiener Algebra $A(\Omega)$

For a function f defined on a domain Ω the Wiener algebra is the space of functions such that the Fourier transform satisfies $\hat{f}(k) \in L^1_k$:

$$A(\mathbb{T}^n) \stackrel{\text{def}}{=} \{f : \textstyle\sum_{k\in\mathbb{Z}^n} |\hat{f}(k)| < \infty\}.$$

It is important to point out that we have the embedding

$$A(\mathbb{T}^n) \subset L^\infty(\mathbb{T}^n).$$

The space $A(\mathbb{T}^n)$ is a Banach algebra:

$$\sum_{k\in\mathbb{Z}^n} |\hat{f} * \hat{g}(k)| \le \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)| \sum_{j\in\mathbb{Z}^n} |\hat{g}(j)|.$$

The non-locality of the Wiener algebra allows us to use this space in ways that we still don't know how for $L^\infty(\mathbb{T}^3)$ in the context of proving estimates for the non-cutoff Boltzmann equation.

In the whole space $\Omega = \mathbb{R}^3$, we define the Wiener Algebra space $A(\Omega) = L^1_\xi$ as the set of tempered distributions on $\mathbb{R}^3$ whose Fourier transform is integrable.

Scale of spaces:

$$L^\infty_x \supset L^1_\xi \supset B^{3/2}_{2,1} \supset H^{3/2+}_x \supset H^2_x.$$

Known examples show that the Wiener algebra L^1_ξ space contains bounded and continuous functions with any arbitrarily low order of regularity, when regularity is measured using the Fourier transform.

Therefore we think of the Wiener algebra as "L^∞_x + a little bit more structure."

Still it appears to be an open problem to characerize the Wiener algebra $A(\Omega)$ in terms of other function spaces.

Example with slowly decaying Fourier transform

Let ϕ be a smooth decaying function with $\hat{\phi} \in [-1, 1]$, $\hat{\phi}(0) = 1$ and $\hat{\phi}(\xi) = 0$ when $|\xi| > 1$. Then for fixed k define

$$\hat{f}_k(\xi) = \sum_{n=1}^{\infty} \frac{\hat{\phi}(\xi - n^k) + \hat{\phi}(\xi + n^k)}{n^2}.$$

Then for fixed ξ, at most one term in $\hat{f}_k$ is non-zero. Further $\hat{f}_k$ has best uniform decay rate for all ξ as

$$\left|\hat{f}_k(\xi)\right| \le \frac{C}{|\xi|^{2/k}}.$$

This function has representation

$$f_k(x) = \phi(x) \sum_{n=1}^{\infty} \frac{2\cos(n^k x)}{n^2}.$$

Thus a continuous function in the Wiener algebra can have arbitrarily slow decay of the Fourier transform.

Recent works using the Wiener Algebra $A(\Omega)$ in PDE

Recent works using $A(\Omega)$ in Fluid Dynamics

- Duchon and Robert (1988) J.D.E.
- Lei and Lin (2011) C. P. A. M.
- Constantin, Córdoba, Gancedo, and S., (2013) J.E.M.S.
- Constantin, Córdoba, Gancedo, Rodríguez-Piazza, and S., (2016) A. J. M.
- Patel and S., (2017) C.P.D.E.
- Gancedo, García-Juárez, Patel, and S., (2019) A. M.

Recent works using $A(\Omega)$ in Materials Science:

- Liu and S. (2019) I.F.B.
- Granero-Belinchón and M. Magliocca (2019) D.C.D.S.-A.
- Ambrose (2019) preprint

New functional space in Kinetic theory

To study well-posedness of the problems in the perturbative framework, we first introduce a function space X_T with $0 < T \leq \infty$, which is a crucial point. For example in a torus $\mathbb{T}^3$, we define

$$X_T \stackrel{\text{def}}{=} L^1_k L^\infty_T L^2_v$$

with norm

$$\|f\|_{X_T} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^3} \sup_{0\leq t\leq T} \|\hat{f}(t,k,\cdot)\|_{L^2_v}.$$

The Fourier transform of $f(x)$ with respect to $x \in \mathbb{T}^3$ is

$$\widehat{f}(k) \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} f(x) e^{-ix\cdot k} dx, \quad k \in \mathbb{Z}^3.$$

This is a Chemin-Lerner type space because the supremum over $0 \leq t \leq T$ is taken before the $L^1_k(\mathbb{Z}^3)$ norm, and this feature plays an important role in controlling the quadratically nonlinear terms.

Norms and function spaces

- Macroscopic projection: $\mathbf{P}f = \{a + b \cdot v + c|v|^2\}\mu^{1/2}$.
- Linearized collision operator satisfies $Lg = L\{\mathbf{I} - \mathbf{P}\}g$.
- Define a "dissipation norm" by $\|\{\mathbf{I} - \mathbf{P}\}f\|^2_{L^2_D} \approx (Lf, f)_{L^2_v}$. (This L^2_D norm is defined precisely in both the non-cutoff Boltzmann case and in the Landau case.)
- Define a velocity weight function: $w_{q,\vartheta}(v) = e^{\frac{q\langle v\rangle^\vartheta}{4}}$ with $\langle v\rangle = \sqrt{1+|v|^2}$ for $q \geq 0$ and $\vartheta \in (0, 2]$.
- And now we introduce the corresponding weighted norms

$$\left\| w_{q,\vartheta} f \right\|_{L^1_k L^\infty_T L^2_v} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^3} \sup_{0\leq t\leq T} \left\| w_{q,\vartheta} \widehat{f}(t,k) \right\|_{L^2_v},$$

$$\left\| w_{q,\vartheta} f \right\|_{L^1_k L^2_T L^2_D} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^3} \left(\int_0^T \left\| w_{q,\vartheta} \widehat{f}(t,k) \right\|^2_{L^2_D} dt \right)^{1/2}.$$

Global theorem on torus without regularity

Theorem 1 (Global existence and uniqueness)

Exists $\epsilon_0 > 0$ *and* $C_0 > 0$ *such that if* $F_0(x,v) = \mu + \mu^{\frac{1}{2}} f_0(x,v) \geq 0$ *and*

$$\|w_{q,\vartheta} f_0\|_{L^1_k L^2_v} \leq \epsilon_0,$$

then there exists a unique global mild solution $f(t,x,v)$*,* $t > 0$*,* $x \in \mathbb{T}^3$*,* $v \in \mathbb{R}^3$ *for the Landau equation or the non-cutoff Boltzmann equation satisfying the non-negativity condition*

$$F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v) \geq 0$$

and for any $T > 0$ *we have the uniform estimate*

$$\|w_{q,\vartheta} f\|_{L^1_k L^\infty_T L^2_v} + \|w_{q,\vartheta} f\|_{L^1_k L^2_T L^2_D} \leq C_0 \|w_{q,\vartheta} f_0\|_{L^1_k L^2_v}.$$

Theorem 2 (Large-time behavior)

Moreover, for fixed $\kappa \in (0,1]$, $\kappa = \kappa(\gamma, s, \vartheta)$, depending on the the Landau or the non-cutoff Boltzmann collision operator as well as $w_{q,\vartheta}$, respectively, there is $\lambda > 0$ such that the solution also enjoys the uniform time decay estimate

$$\|f(t)\|_{L^1_k L^2_v} \lesssim e^{-\lambda t^{\kappa}} \|w_{q,\vartheta} f_0\|_{L^1_k L^2_v},$$

for any $t \geq 0$.

Remark

Recent numerical study on the possible sharp $2/3$ rate of large time decay for the Landau equation with Coulomb interaction ($\gamma = -3$) in Bobylev-Gamba-Zhang (2017) J.S.P.; Pennie-Gamba (2019) (arXiv:1910.03110) also obtains the same behavior using a completely different numerical method; **We obtain this decay rate, $\kappa = \frac{2}{3}$ above, when $\gamma = -3$, $s = 1$, and $\vartheta = 2$.**

Theorem 3 (Propagation of spatial regularity: $\langle k\rangle^m$)

Let all the conditions in Theorem 1 be satisfied, then for any integer $m \geq 0$, there is an $\epsilon_m > 0$ and a $C_m > 0$ such that if

$$\sum_{k\in\mathbb{Z}^3} \langle k\rangle^m \|w_{q,\vartheta}\hat{f}_0(k)\|_{L^2_v} \overset{def}{=} \|w_{q,\vartheta} f_0\|_{L^1_{k,m}L^2_v} \leq \epsilon_m,$$

then the solution $f(t,x,v)$ established in Theorem 1 satisfies

$$\sum_{k\in\mathbb{Z}^3} \langle k\rangle^m \sup_{0\leq t\leq T} \|w_{q,\vartheta}\hat{f}(t,k)\|_{L^2_v} + \sum_{k\in\mathbb{Z}^3} \langle k\rangle^m \|w_{q,\vartheta}\hat{f}(t,k)\|_{L^2_T L^2_D} \leq C_m \|w_{q,\vartheta} f_0\|_{L^1_{k,m}L^2_v}$$

for any $T > 0$.

Remark: We expect that this theorem may be improved to only require the smallness condition from Theorem 1.

On $\mathbb{T}^3$, these solutions have C^∞ smoothing

- Due to the L^∞_x embedding, these solutions have bounded (in space and time) mass, energy, and entropy (in velocity).
- Recent work of Imbert-Silvestre (2019, arXiv:1909.12729), also their review article (2020, arXiv:2005.02997), proves the $C^\infty_{t,x,v}$ smoothing effect for the non-cutoff Boltzmann equation with the hard potentials $\gamma + 2s \geq 0$ conditional to the macroscopic quantities (mass, energy, and entropy) being bounded in space and time. Plus some minor velocity bounds.
- Our solutions satisfy the assumptions of their theorem when $\gamma + 2s \geq 0$ and thus experience the $C^\infty_{t,x,v}$ smoothing effect. (I am ignoring some technical issues when $\gamma \leq 0$ for simplicity.)
- These currently may be the lowest regularity global in time solutions to the non-cutoff Boltzmann equation that are known to experience the $C^\infty_{t,x,v}$ regularization.

- In summary our theorems establish global existence, uniqueness, non-negativity, large time decay rates to equilibrium, and propagation of spatial regularity.
- The theorems above hold for the non-cutoff Boltzmann equation and the Landau equation in the space $X_T \stackrel{\text{def}}{=} L^1_k L^\infty_T L^2_v$ without using regularity in the torus $\mathbb{T}^3$ with periodic boundary conditions.
- These theorems will also hold in the whole space $\mathbb{R}^3$ with suitable modifications to the time decay rate without regularity.
- We have analogous theorems in the space $L^1_k L^\infty_T L^2_{x_1} L^2_v$ using one spatial derivative in the finite channel

$$\Omega = I \times \mathbb{T}^2 = \left\{x = (x_1, \bar{x}),\ x_1 \in (-1, 1), \bar{x} := (x_2, x_3) \in \mathbb{T}^2\right\}.$$

with the inflow boundary conditions on $(-1, 1)$.

- We also have analogous theorems in the finite channel with the specular reflection boundary condition, further using a standard symmetry condition, $f_0(-x_1, -v_1) = f_0(x_1, v_1)$, on the initial data (which allows us to rule out the singularity at the boundary).

High Level Strategy of Proof: Illustrative toy model

Toy model of the energy method machinery for the heat equation: Suppose u is a periodic solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

We can multiply both sides by u and integrate with respect to x:

$$\int_0^1 u\frac{\partial u}{\partial t}dx = \int_0^1 u\frac{\partial^2 u}{\partial x^2}dx \qquad \Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u(x,t))^2 dx = -\int_0^1 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx$$

From here we see that square-integrable solutions of the heat equation have norms that decrease in time. We can even add extra nonlinear terms to this equation as long as they can be shown to be "small" relative to the square integral of $\frac{\partial u}{\partial x}$.

A key step in the proof

$$\sum_{k\in\mathbb{Z}^3}\sup_{0\le t\le T}\|\hat{f}(t,k,\cdot)\|_{L^2_v}+\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\|\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k,\cdot)\|^2_{L^2_D}\,dt\right)^{1/2}$$
$$\lesssim\|f_0\|_{L^1_kL^2_v}+\|f\|_{L^1_kL^\infty_TL^2_v}\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\|\hat{f}(t,k,\cdot)\|^2_{L^2_D}\,dt\right)^{1/2}$$

This inequality for solutions indicates that as long as one can further appropriately estimate the macroscopic dissipation:

$$\sum_{k\in\mathbb{Z}^3}\|\mathbf{P}\hat{f}(k)\|_{L^2_TL^2_D}\approx\sum_{k\in\mathbb{Z}^3}\|\widehat{(a,b,c)}(k)\|^2_{L^2_T}$$

We can then obtain a global in time uniform estimate on the solution under the smallness assumption on $\|f\|_{L^1_kL^\infty_TL^2_v}$ that can be closed using a continuity argument provided $\|f_0\|_{L^1_kL^2_v}$ is initially suitably small.

Proof of the global inequality

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f),$$

Taking the Fourier transform in $x \in \mathbb{T}^3$ we obtain

$$\partial_t \hat{f}(t,k,v) + iv \cdot k\hat{f}(t,k,v) + L\hat{f}(t,k,v) = \hat{\Gamma}(\hat{f},\hat{f})(t,k,v).$$

Here, for brevity $\hat{\Gamma}$ indicates convolutions in k:

$$\hat{\Gamma}(\hat{f},\hat{g})(k,v) =$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\sigma)\mu^{1/2}(u)\left([\hat{f}(u') * \hat{g}(v')](k) - [\hat{f}(u) * \hat{g}(v)](k)\right) d\sigma du,$$

where the convolutions are taken with respect to $k \in \mathbb{Z}^3$:

$$[\hat{f}(u') * \hat{g}(v')](k) \stackrel{\text{def}}{=} \sum_{l \in \mathbb{Z}^3} \hat{f}(k-l,u')\hat{g}(l,v').$$

We use the coercivity estimate for L (for $\delta_0 > 0$):

$$(L\hat{f}, \hat{f})_{L^2_v} \geq \delta_0 \|\{\mathbf{I} - \mathbf{P}\}\hat{f}\|^2_{L^2_D}$$

and integrate over L^2_v to obtain

$$\frac{1}{2}\|\hat{f}(t,k,\cdot)\|^2_{L^2_v} + \delta_0 \int_0^t \|\{\mathbf{I} - \mathbf{P}\}\hat{f}\|^2_{L^2_D} d\tau \leq \frac{1}{2}\|\hat{f}_0(k,\cdot)\|^2_{L^2_v} + \int_0^t \left|\mathrm{Re}(\hat{\Gamma}(\hat{f},\hat{f}),\hat{f})_{L^2_v}\right| d\tau.$$

Taking the square root on both sides and using the elementary inequality $\frac{1}{\sqrt{2}}(A+B) \leq \sqrt{A^2+B^2} \leq A+B$, we further have

$$\frac{1}{\sqrt{2}}\|\hat{f}(t,k,\cdot)\|_{L^2_v} + \sqrt{\delta_0}\left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\}\hat{f}(\tau,k,\cdot)\|^2_{L^2_D} d\tau\right)^{1/2} \leq \|\hat{f}_0(k,\cdot)\|_{L^2_v} + \sqrt{2}\left(\int_0^t \left|\mathrm{Re}(\hat{\Gamma}(\hat{f},\hat{f}),\hat{f})_{L^2_v}\right| d\tau\right)^{1/2}.$$

So, we have derived, for any $0 \le t \le T$ and $k \in \mathbb{Z}^3$, that we

$$\|\hat{f}(t,k,\cdot)\|_{L^2_v} + \left(\int_0^t \|\{\mathbf{I}-\mathbf{P}\}\hat{f}(\tau,k,\cdot)\|^2_{L^2_D}\,d\tau\right)^{1/2} \le C_0\left\{\|\hat{f}_0(k,\cdot)\|_{L^2_v} + \left(\int_0^t \left|\mathrm{Re}(\hat{\Gamma}(\hat{f},\hat{f}),\hat{f})_{L^2_v}\right| d\tau\right)^{1/2}\right\},$$

with $C_0 > 0$. Moreover, taking $\sup_{0\le t\le T}$ on both sides above and then summing the resulting inequality over $k \in \mathbb{Z}^3$, we have

$$\sum_{k\in\mathbb{Z}^3}\sup_{0\le t\le T}\|\hat{f}(t,k,\cdot)\|_{L^2_v} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T \|\{\mathbf{I}-\mathbf{P}\}\hat{f}(\tau,k,\cdot)\|^2_{L^2_D}\,dt\right)^{1/2} \le C_0\left\{\|\hat{f}_0\|_{L^1_k L^2_v} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\left(\hat{\Gamma}(\hat{f},\hat{f}),\hat{f}\right)_{L^2_v}\right| dt\right)^{1/2}\right\}.$$

Q: How do you estimate the non-linear term?

Trilinear estimates

By the definition of $\hat{\Gamma}(\cdot,\cdot)$ as well as Fubini's theorem:

$$
\begin{aligned}
&(\hat{\Gamma}(\hat{f},\hat{g})(k),\hat{h}(k))_{L^2_v}\\
&\stackrel{\text{def}}{=} \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_u}\int_{\mathbb{S}^2} B(v-u,\sigma)\mu^{1/2}(u)\\
&\qquad\qquad \times\left\{[\hat{f}(u')*\hat{g}(v')](k)-[\hat{f}(u)*\hat{g}(v)](k)\right\}\bar{\hat{h}}(v,k)d\sigma du dv\\
&=\sum_{l\in\mathbb{Z}^3}\int\int\int B\mu^{1/2}\left\{\hat{f}(k-l,u')\hat{g}(l,v')-\hat{f}(k-l,u)\hat{g}(l,v)\right\}\bar{\hat{h}}(v,k)\\
&=\sum_{l\in\mathbb{Z}^3}\int_{\mathbb{R}^3_v}\Gamma(\hat{f}(k-l),\hat{g}(l))(v)\bar{\hat{h}}(k,v)dv.
\end{aligned}
$$

Therefore we conclude

$$
\left|(\hat{\Gamma}(\hat{f},\hat{g})(k),\hat{h}(k))_{L^2_v}\right|\le\sum_{l\in\mathbb{Z}^3}\left|(\Gamma(\hat{f}(k-l),\hat{g}(l)),\hat{h}(k))_{L^2_v}\right|.
$$

Trilinear estimates

- As above we have

$$\left|(\hat{\Gamma}(\hat{f},\hat{g})(k),\hat{h}(k))_{L^2_v}\right| \leq \sum_{l\in\mathbb{Z}^3}\left|(\Gamma(\hat{f}(k-l),\hat{g}(l)),\hat{h}(k))_{L^2_v}\right|.$$

- Now we can directly use the old estimates

$$|(\Gamma(f,g),h)_{L^2_v}| \lesssim \|f\|_{L^2_v}\|g\|_{L^2_D}\|h\|_{L^2_D}.$$

  E.g. Guo (2002, Landau), AMUXY (2011, non-cutoff Boltzmann), Gressman-S. (2011, non-cutoff Boltzmann), S.-Zhu (2013, improvement for Landau)

- From both of those together, we obtain the trilinear estimate

$$\left|\left(\hat{\Gamma}(\hat{f},\hat{g})(k),\hat{h}(k)\right)_{L^2_v}\right| \lesssim \sum_{l\in\mathbb{Z}^3}\|\hat{f}(k-l)\|_{L^2_v}\|\hat{g}(l)\|_{L^2_D}\|\hat{h}(k)\|_{L^2_D}$$

Trilinear estimates

$$\left| \left(\hat{\Gamma}(\hat{f}, \hat{g})(k), \hat{h}(k) \right)_{L^2_v} \right| \lesssim \sum_{l \in \mathbb{Z}^3} \|\hat{f}(k-l)\|_{L^2_v} \|\hat{g}(l)\|_{L^2_D} \|\hat{h}(k)\|_{L^2_D}$$

Then we estimate the non-linear term as follows for any small $\epsilon > 0$:

$$\sum_{k \in \mathbb{Z}^3} \left(\int_0^T \left| \left(\hat{\Gamma}(\hat{f}, \hat{g}), \hat{h} \right)_{L^2_v} (k) \right| dt \right)^{1/2}$$

$$\leq \sum_{k \in \mathbb{Z}^3} \left(\int_0^T \sum_{l \in \mathbb{Z}^3} \|\hat{f}(k-l)\|_{L^2_v} \|\hat{g}(l)\|_{L^2_D} \|\hat{h}(k)\|_{L^2_D} dt \right)^{1/2}$$

$$\leq \cdots \leq C_\epsilon \|f\|_{L^1_k L^\infty_T L^2_v} \|g\|_{L^1_k L^2_T L^2_D} + \epsilon \|h\|_{L^1_k L^2_T L^2_D}.$$

The non-locality of the Fourier transform helps a lot where it is still open to prove an estimate like this using instead L^∞_x.

Trilinear estimates

$$\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\left|\left(\hat{\Gamma}(\hat{f},\hat{g}),\hat{h}\right)_{L^2_v}\right|dt\right)^{1/2}\leq C_\epsilon\|f\|_{L^1_kL^\infty_TL^2_v}\|g\|_{L^1_kL^2_TL^2_D}+\epsilon\|h\|_{L^1_kL^2_TL^2_D}$$

We also have the following identity:

$$\left(\hat{\Gamma}(\hat{f},\hat{f}),\hat{f}\right)_{L^2_v}=\left(\hat{\Gamma}(\hat{f},\hat{f}),\{\mathbf{I}-\mathbf{P}\}\hat{f}\right)_{L^2_v}.$$

Using this for $\hat{h}=\{\mathbf{I}-\mathbf{P}\}\hat{f}$ allows us to directly establish:

$$\sum_{k\in\mathbb{Z}^3}\sup_{0\leq t\leq T}\|\hat{f}(t,k,\cdot)\|_{L^2_v}+\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\|\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k,\cdot)\|^2_{L^2_D}\,dt\right)^{1/2}$$
$$\lesssim\|f_0\|_{L^1_kL^2_v}+\|f\|_{L^1_kL^\infty_TL^2_v}\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\|\hat{f}(t,k,\cdot)\|^2_{L^2_D}\,dt\right)^{1/2}$$

This is the main uniform global inequality in our existence proof.

Thank you!

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Global mild solutions of the Landau and non-cutoff Boltzmann equations. Comm. Pure Appl. Math.
pages 1–89, (2020), arXiv:1904.12086, doi:10.1002/cpa.21920.