On Yau's conjecture in Euclidean domains

Eugenia Malinnikova (Stanford)

joint work with A. Logunov, N. Nadirashvili, F. Nazarov

PDE Seminar via Zoom

October 1, 2020, ShanghaiTech University

History of Yau's conjecture

In 1970s Yau formulated the following problem

M is a Riemannian manifold, u_λ is an eigenfunction of the Laplace-Beltrami operato

$$\Delta_M u_\lambda + \lambda u_\lambda = 0 \implies c_M\sqrt{\lambda} \underset{(*)}{\le} \mathcal{H}^{n-1}(u_\lambda = 0) \underset{(**)}{\le} C_M\sqrt{\lambda}$$

- Yau, Brunning 1970s $(*)$ when $n=2$
- Donnelly and Fefferman 1980s $(*)$ & $(**)$ if metric is real analytic // 1990s $(*)$ & $(**)$ for manifolds with boundary, metric and boundary are analytic
- Logunov 2016, $(*)$ for smooth manifolds
  weak $(**)$ $= C_M\lambda^a$, $a=a(n)$ for smooth manifolds

Main result

We consider $\Omega \subset \mathbb{R}^n$ (manifold with analytic metric) and Dirichlet-Laplace eigenfunctions.

$$\begin{cases} \Delta u_\lambda + \lambda u_\lambda = 0 \\ u_\lambda = 0 \text{ on } \partial\Omega \end{cases} \Rightarrow \quad \mathcal{H}^{n-1}(u_\lambda = 0) \geq c_\Omega \sqrt{\lambda}$$

by results of Donnelly and Fefferman

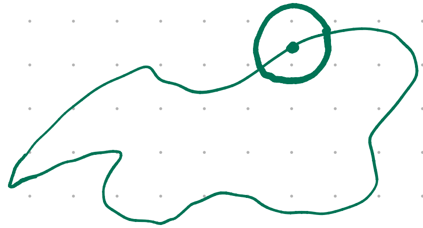
Theorem (Logunov, M, Nadirashvili, Nazarov)

Suppose that $\partial\Omega$ is of class $C^{1,\text{Dini}}$, then

$$\mathcal{H}^{n-1}(u_\lambda = 0) \leq C_\Omega \sqrt{\lambda}$$

Question Is this true in Lipschitz domains?

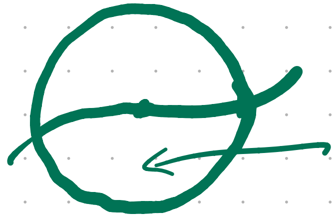
Regularity of $\partial\Omega$



$y \in \partial\Omega$, r, ω fixed

$$\Omega \cap B(y,r) = \{ (x', x_n) \in \mathbb{R}^n : \quad x_n < \varphi(x') \}$$

$$\nabla\varphi(y) = 0, \quad |\nabla\varphi(x) - \nabla\varphi(y)| \leq \omega(|x-y|)$$



$\Omega \cap B(y,r)$

ω is a regular Dini modulus of continuity

- $\omega(s) < \omega(t)$, $\quad s < t$
- $\dfrac{\omega(s)}{s} > \dfrac{\omega(t)}{t}$, $\quad s < t$
- $\displaystyle\int_0^r \frac{\omega(t)}{t}\,dt < \infty$, $\qquad \tilde{\omega}(s) = \int_0^s \frac{\omega(t)}{t}\,dt$

A slightly related uniqueness problem for harmonic functions

$\Delta h = 0$, $D \subset \partial\Omega$ open, $h = 0$ on D
and $E \subset D$, $\mathcal{H}^{n-1}(E) > 0$, $h_n = 0$ on E $\overset{?}{\Longrightarrow} h = 0$

$C^{1,1}$ domains	F.-H. Lin	1991
Convex Lipshitz	V. Adolfsson, L. Escauriaza, C. E. Kenig	1995
$C^{1,\alpha}$ domains	V. Adolfsson & L. Escauriaza	1997
$C^{1,\mathrm{Dini}}$	I. Kukavica, K. Nyström	1998
Lipshitz with small constant	X. Tolsa	2020

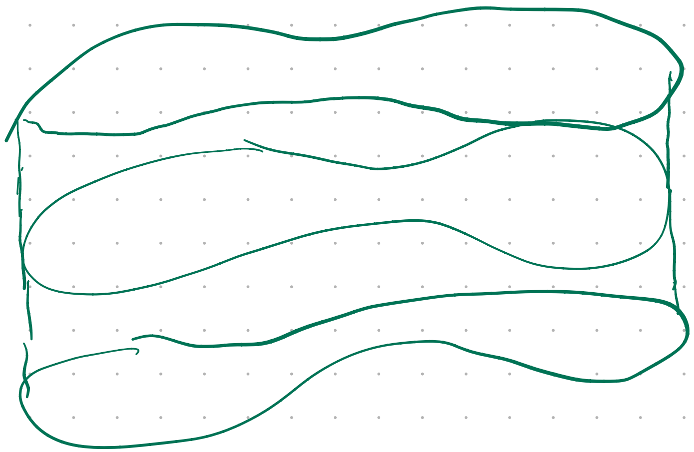
One of the common tools is analysis of doubling indices introduced by Logunov

From eigenfunctions to harmonic functions

$$h(x,t) = e^{t\sqrt{\lambda}}\, u(x) \quad \text{in} \quad \Omega \times (-1,1) \qquad \Delta u + \lambda u = 0$$

$$Z(h) = Z(u) \times (-1,1) \quad , \quad Z(h) = \{h = 0\},$$

$$h = 0 \quad \text{on} \quad \partial\Omega \times (-1,1) \qquad \Delta h = 0$$



$$\Omega \times (-1,1) \subset \mathbb{R}^{n+1}$$

Doubling indices for harmonic functions

$y \in \overline{\Omega}$ $\quad N_h(y,r) = \log \frac{\int_{B(y,2r)} h^2}{\int_{B(y,r)} h^2}$

Monotonicity of the doubling indices

$N_h(y,r)e^{Cr} \nearrow$ as $r \nearrow$

$C^{1,\text{Dini}}$ domain with $h=0$ on $\partial\Omega \cap B(y,R)$

(Kukavica, Nystrom)

$r_1 < r_2$

$N_h(y,r_1) \le$

$\le C\, N_h(y,r_2)$

$C(r_2) \to 1,\ r_2 \to 0.$

Doubling for the lift function

Donnelly & Fefferman $\quad h(x,t) = e^{\sqrt{\lambda}t}u(x),\ \Delta u + \lambda u = 0$

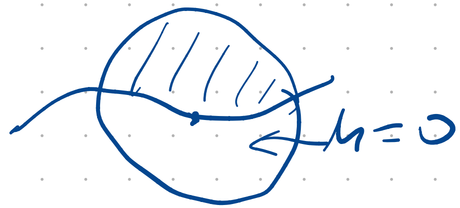
$\Rightarrow N_h(y,r) \le C_\Omega \sqrt{\lambda}$

Reformulation of the main result

Theorem $\Delta h = 0$ in $\Omega \subset \mathbb{R}^{n+1}$, $\partial\Omega \cap B(y, 2r)$ is $C^{1,\text{Dini}}$

$h = 0$ on $\partial\Omega \cap B(y, 2r)$ $\quad y \in \partial\Omega$ $\quad h=0 \cap B \cap \Omega$

Then $\mathcal{H}^n(\{h=0\} \cap \Omega \cap B(y,r)) \le C r^n \underline{N_h(y,r)}$



Tool: Quantitative Cauchy uniqueness

$\partial\Omega \cap B(y, 2r)$ is Dini, $\Delta h = 0$ in Ω

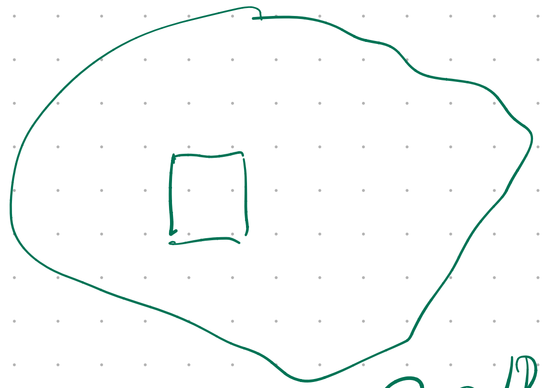
$|h| \le 1$ in $\Omega \cap B(y, 2r)$ and $|h| \le \varepsilon$ on $\partial\Omega \cap B(y, 2r)$

$|\nabla h| \le \frac{1}{r}$ in $\Omega \cap B(y, 2r)$ $\quad$ $|\nabla h| \le \frac{\varepsilon}{r}$

$$\Rightarrow |h| \le C \varepsilon^{\alpha}$$

in $B(y, r)$

Estimate of Whitney cubes



$\Omega \subset \mathbb{R}^{n+1}$

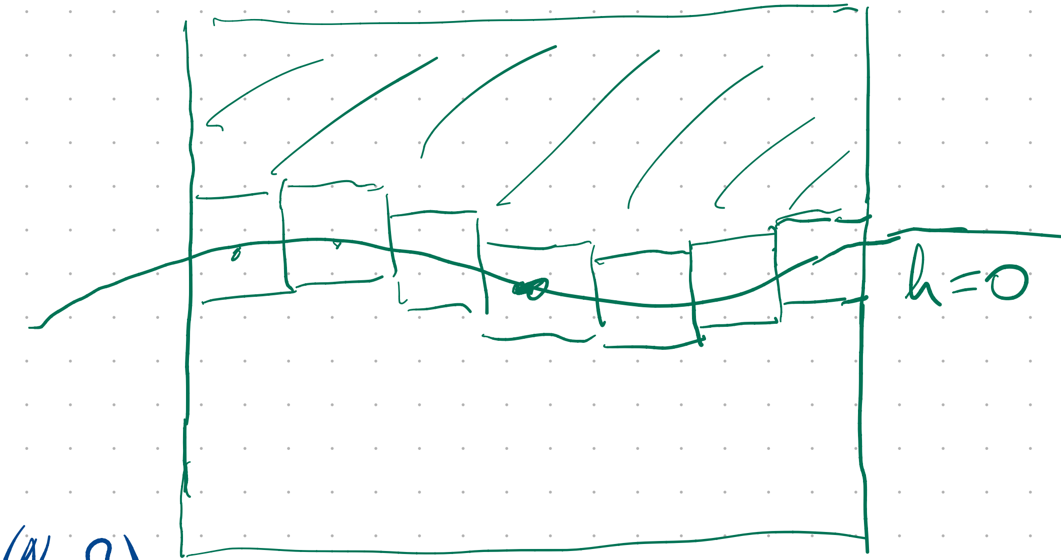
$Q \subset \Omega, \quad \operatorname{dist}(Q, \partial\Omega) \geq c\,\ell(Q)$

Then analytic results of Donnelly and Fefferman

$$\Rightarrow \quad \mathcal{H}^n(\{h=0\} \cap Q) \leq C\, N_h^*(Q) := \sup_{\substack{y \in Q \cap \bar{\Omega} \\ r < \operatorname{diam}(Q)}} N_h(y,r)$$

The question is to estimate the size of $\{h=0\}$ near $\partial\Omega$.

Induction on scales



$$N_h^*(Q) = \sup_{\substack{y \in Q \cap \overline{\Omega} \\ r < \operatorname{diam}(Q)}} N_h(y, r)$$

Lemma 1 $N_h^*(Q) \le N \Rightarrow \exists\, k = k(N, \Omega)$ such that if $Q \cap \partial\Omega \subset \bigcup_1^k c_j$ are equal cubes $\Rightarrow$ for one j

$$(\Omega \cap c_j) \cap \{h = 0\} = \emptyset$$

Iterations $\Rightarrow$ $\mathcal{H}^n(h=0) \le F(N)$ $\quad (\Delta)$

Lemma 2 $\exists\, l = l(\Omega)$ s.t. if $N_h^*(Q) > N_0$ and $Q \cap \partial\Omega \subset \bigcup_1^l q$ are equal cubes $\Rightarrow$ for one j

$$N_h^*(q_j) \le \frac{N_h^*(Q)}{2}$$

Iterations $+ (\Delta)$ $\Rightarrow$ $\mathcal{H}^n(h=0) \le CN$.

$\nabla h \in C^1(\overline{\Omega})$

$|\nabla h(x) - \nabla h(y)| \le$

$\le C\, \tilde{\omega}(|x-y|) \cdot \sup_{2Q \cap \Omega} |h|$

$x, y \in Q \cap \Omega$

M is a surface, $\dim M = 2$, $\Delta u + \lambda u = 0$

Donnelly & Fefferman 1990 $\mathcal{H}^1\{u=0\} \le C\lambda^{3/4}$

(Carleman inequality)

Dong 1992

$\Delta u + \lambda u = 0$, $\Delta \log(|\nabla u|^2 + \frac{\lambda}{2} u^2) \ge -\lambda - K_M$.

Logunov, M 2016 there is $\varepsilon > 0$ $\mathcal{H}^1(u=0) \le C\lambda^{3/4-\varepsilon}$

? Yau's conjecture in dimension two $\mathcal{H}^1(u=0) \le C\sqrt{\lambda}$