Instability and non-uniqueness of Leray solutions of the forced Navier-Stokes equations, Part 3 [1]

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[1] Joint with D. Albritton and M. Colombo

Main result

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = f \\ \operatorname{div} u = 0 \\ u(\cdot, 0) = u_0 \end{cases} \quad \text{on } \mathbb{R}^3 \times [0, T] \qquad \text{(NS)}$$

Theorem (Albritton-B.-Colombo '21, '22)

Let Ω be $\mathbb{R}^3$, a smooth bounded domain, or $\mathbb{T}^3$. Then, there exist u and $\bar{u}$, two distinct suitable Leray-Hopf solutions to (NS) with identical body force $f \in L^1_t L^2_x$ and $u(\cdot, 0) = \bar{u}(\cdot, 0) = 0$. When Ω is a bounded domain, u and $\bar{u}$ satisfy no-slip boundary conditions and f is supported far away from the boundary.

Previous lectures

Theorem (Albritton-B.-Colombo '21)

There exist two distinct suitable Leray-Hopf solutions to (NS) with identical body force $f \in L^1_t L^2_x$ and $u(\cdot,0) = \bar{u}(\cdot,0) = 0$.

- Linear instability: There exists $\bar{U} \in C^\infty_c$ such that

$$\mathcal{L}_{\mathrm{ss}} : D(\mathcal{L}_{\mathrm{ss}}) \subset L^2_\sigma \to L^2_\sigma$$

  has a maximal unstable eigenvalue.

- Nonlinear instability: The unstable eigenvalue can be perturbed to $\bar{U}$, an unstable trajectory for (ssNS). In standard variables, $u(x,t) = \frac{1}{\sqrt{t}} U(\xi)$ provides a second solution to (NS) with body force f and $u(0,\cdot) = 0$.

Theorem (Linear instability)

There exists a divergence-free vector field $\bar{U} \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ s.t. $\mathcal{L}_{ss} : D(\mathcal{L}_{ss}) \subset L^2_\sigma(\mathbb{R}^3;\mathbb{R}^3) \to L^2_\sigma(\mathbb{R}^3;\mathbb{R}^3)$

$$-\mathcal{L}_{\mathrm{ss}} V = -\frac{1}{2}(1 + \xi \cdot \nabla) V - \Delta V + \mathbb{P}(\bar{U} \cdot \nabla V + V \cdot \nabla \bar{U})$$

has a maximal unstable eigenvalue.

Theorem (Nonlinear instability)

Let $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda > 0$, be a maximal unstable eigenvalue of $\bar{U}$ with eigenfunction $\eta \in H^k$ for all $k \in \mathbb{N}$. Set $U^{\mathrm{lin}}(\xi,\tau) = \mathrm{Re}(e^{\lambda\tau}\eta(\xi))$. There exist $T \in \mathbb{R}$ and a div-free vector field $U^{\mathrm{per}} : \mathbb{R}^3 \times (-\infty, T) \to \mathbb{R}^3$ such that

- *Regularity and decay:*

$$\|U^{\mathrm{per}}(\cdot,\tau)\|_{H^k} \lesssim e^{2a\tau}, \quad \tau \le T,\ k \ge 0$$

- $U := \bar{U} + U^{\mathrm{lin}} + U^{\mathrm{per}}$ *solves*

$$\partial_\tau U - \frac{1}{2}(1 + \xi \cdot \nabla) U - \Delta U + U \cdot \nabla U + \nabla P = F \qquad \text{(ssNS)}$$

Construction of a linear unstable backgrounds

Theorem (Albritton-B.-Colombo)

There exists a divergence-free vector field $\bar{U} \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ s.t. the linear operator $\mathcal{L}_{ss} : D(\mathcal{L}_{ss}) \subset L^2_\sigma(\mathbb{R}^3;\mathbb{R}^3) \to L^2_\sigma(\mathbb{R}^3;\mathbb{R}^3)$

$$D(\mathcal{L}_{\rm ss}) := \{V \in L^2_\sigma \,:\, V \in H^2(\mathbb{R}^3)\,,\, \xi \cdot \nabla U \in L^2(\mathbb{R}^3)\}\,,$$

$$-\mathcal{L}_{\rm ss} U = -\frac{1}{2}(1 + \xi \cdot \nabla)U - \Delta U + \mathbb{P}(\bar{U} \cdot \nabla U + U \cdot \nabla \bar{U})\,,$$

has an unstable eigenvalue.

Definition (Linear Instability)

We say that $\mathcal{L}_{\rm ss} : D(\mathcal{L}_{\rm ss}) \subset L^2_\sigma \to L^2_\sigma$ has an unstable eigenvalue if there exist

- $\lambda \in \mathbb{C}$ with $a := \mathrm{Re}\lambda > 0$
- $\eta \in H^k(\mathbb{R}^3;\mathbb{R}^3)$ for any $k > 0$, with div $\eta = 0$

such that

$$\mathcal{L}_{\rm ss}\eta = \lambda\eta\,.$$

Gluing non-unique Navier-Stokes solutions

Theorem (Albritton-B.-Colombo '22)

Let Ω be a smooth bounded domain, or $\mathbb{T}^3$. Then, there exist u and $\bar{u}$, two distinct suitable Leray-Hopf solutions to (NS) with identical body force $f \in L^1_t L^2_x$, initial condition $u(\cdot, 0) = \bar{u}(\cdot, 0) = 0$, and no-slip boundary conditions.

Plan of the talk

- Linear instability and Vortex ring construction
- Gluing non-unique Navier-Stokes solutions
- Open questions

Linear instability and vortex ring construction

Construction of a linear unstable backgrounds

Theorem (Albritton-B.-Colombo)

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such that

$$\mathcal{L}_{\rm ss}\eta = \lambda \eta\, .$$

2D Instability: Vishik's theorem

$$L^2_m(\mathbb{R}^2) := \bigotimes_{k=1}^{\infty} \{\omega \in L^2(\mathbb{R}^2) \,:\, \omega = f(r)e^{ikm\theta}\}\,.$$

Theorem (Vishik '18, ABCDGJK'21)

There exist $m \geq 2$ and a smooth decaying vortex

$$\bar{u}(x) = \zeta(r)x^{\perp}\,, \qquad \bar{\omega}(x) = g(r)\,,$$

such that $\mathcal{L} : D(\mathcal{L}) \subset L^2_m(\mathbb{R}^2) \to L^2_m(\mathbb{R}^2)$,

$$\mathcal{L}\omega = -\zeta(r)\partial_\theta\omega - (\mathrm{BS}_2[\omega] \cdot e_r)g'(r)\,.$$

has an unstable eigenvalue.

How to build an unstable 3D-vortex ring

- Step 1: We truncate $\bar{u}$ to get an unstable, compactly supported vortex
- Step 2: We use the truncate vortex as a radial profile of $3D$-Axisymmetric-no-swirl velocity field
- Step 3: We show that the vortex ring inherits the instability of Vishik's vortex

Step 1: Compactly supported unstable vortex

Let $\lambda_\infty \in \mathbb{C} \cap \{\mathrm{Re} > 0\}$ be an unstable eigenvalue of

$$\mathcal{L}\omega = -\zeta(r)\partial_\theta\omega - (\mathrm{BS}_2[\omega] \cdot e_r)g'(r)\,,$$

where $u = \zeta(r)x^\perp$, $\omega = g(r)$ is Vishik's unstable vortex.

Proposition (Truncated unstable vortex)

For any $\varepsilon \in (0, \mathrm{Re}\lambda_\infty)$, if $R \geq R(\varepsilon, u)$ the following holds. The linear operator

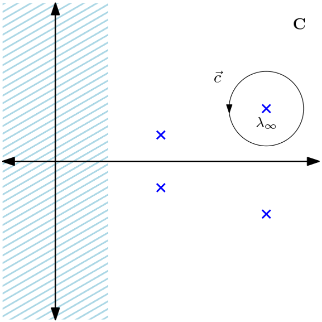
$$\mathcal{L}_R : D(\mathcal{L}_R) \subset L^2_m(\mathbb{R}^2) \to L^2_m(\mathbb{R}^2)$$

associated with the truncated vortex

$$\bar{u}_R(x) = \zeta(r)\chi_R(r)x^\perp\,, \qquad \bar{\omega}(x) = \mathrm{curl}\,\bar{u}_R(x) = g_R(r)\,,$$

admits an unstable eigenvalue λ_R with $|\lambda_R - \lambda_\infty| \leq \varepsilon$.

Spectral projection



- Spectral projection: Let $\vec{c}$ be a simple closed curve in $\mathbb{C}$,

$$\mathrm{Pr}_{\vec{c}}(\mathcal{L}) = \frac{1}{2\pi i}\int_{\vec{c}} R(\lambda, \mathcal{L})\ \mathrm{d}\lambda$$

is the spectral projection in the region enclosed by $\vec{c}$.

- Resolvent:

$$R(\lambda, \mathcal{L}) = (\lambda - \mathcal{L})^{-1}\,.$$

- Let $\vec{c}$ a simple closed curve enclosing λ_∞, we need to prove that

$$\mathrm{Pr}_{\vec{c}}(\mathcal{L}_R) \to \mathrm{Pr}_{\vec{c}}(\mathcal{L})\,, \quad \text{as } R \to \infty\,.$$

- It is enough to show that

$$R(\lambda, \mathcal{L}_R) \to R(\lambda, \mathcal{L})\,, \quad \text{as } R \to \infty, \text{ for any } \lambda \in \vec{c}\,.$$

Spectral perturbation argument

- We write

$$\begin{aligned}\lambda - \mathcal{L}_R &= \lambda - \mathcal{L} + (\mathcal{L} - \mathcal{L}_R) \\ &= (\lambda - \mathcal{L})[I + R(\lambda, \mathcal{L})(\mathcal{L} - \mathcal{L}_R)] \\ &= (\lambda - \mathcal{L}) + o(1)\,, \quad \text{as } R \to \infty\,.\end{aligned}$$

  Where we used that $\mathcal{L}_R \to \mathcal{L}$ in a suitable sense.

- It is immediate to conclude that

$$R(\lambda, \mathcal{L}_R) \to R(\lambda, \mathcal{L})\,, \quad \text{as } R \to \infty, \text{ for any } \lambda \in \vec{c}\,.$$

Step 2: Lifting the truncated vortex to a vortex ring

Let $u(x) = \zeta(r)x^\perp$, $\omega = \operatorname{curl} u = g(r)$ be a vortex. Let

$$\mathcal{L}\omega = -\zeta(r)\partial_\theta \omega - (\mathrm{BS}_2[\omega] \cdot e_r)g'(r)\,,$$

be the associated linearized operator. We assume that

- Compactly supported: $\operatorname{supp} u, \operatorname{supp} \omega \subset B_1(0)$
- Instability: There exist $\lambda \in \mathbb{C} \cap \{\mathrm{Re} > 0\}$ and $\rho \in L^2(\mathbb{R}^2)$ such that $\mathcal{L}\rho = \lambda\rho$.

Remark

$\operatorname{supp} \rho \subset B_1(0)$ and λ is an isolated eigenvalue.

Axisymmetric-no-swirl structure

$$x = (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3$$

Axisymmetric-no-swirl vector fields:

$$U = U^r(r,z)e_r + U^z(r,z)e_z\,,$$

$$\mathrm{curl} U = -(\partial_z U^r - \partial_r U^z)e_\theta = -\Omega(r,z)e_\theta\,.$$

- We assume $\bar{U} = \bar{U}^r(r,z)e_r + \bar{U}^z(r,z)e_z$.
- The space

$$L^2_{\mathrm{ans}}(\mathbb{R}^3) := \{U \in L^2_\sigma(\mathbb{R}^3)\ :\ U \text{ is axisymmetric-no-swirl}\}$$

is invariant under the action of

$$-\mathcal{L}_{\mathrm{ss}} U = -\frac{1}{2}(1 + \xi\cdot\nabla)U - \Delta U + \mathbb{P}(\bar{U}\cdot\nabla U + U\cdot\nabla\bar{U})\,.$$

The 3D vortex ring

- Let u, $\omega = \mathrm{curl}\, u$ be the unstable compactly supported vortex,

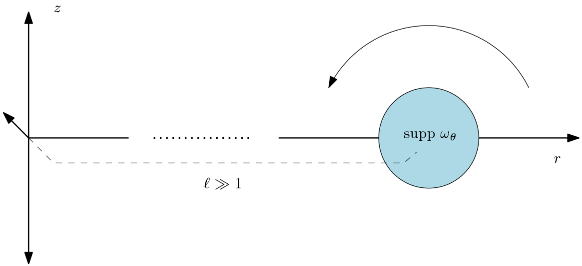
$$u(x) = u(x_1, x_2) = (u^1(x_1, x_2), u^2(x_1, x_2)), \quad x = (x_1, x_2) \in \mathbb{R}^2$$

- Let $\ell \gg 1$. We set

$$\tilde{U}(r, z) = u^1(r - \ell, z)e_r + u^2(r - \ell, z)e_z$$

Notice that

$$\mathrm{curl}\, \tilde{U} = -\omega(r - \ell, z)e_\theta\,.$$



The 3D vortex ring

- Divergence:

$$\operatorname{div} \tilde{U} = \left(\partial_r + \frac{1}{r}\right) u^1 + \partial_z u^2 = \frac{1}{r} u^1 \ll 1 \quad \text{if } \ell \gg 1$$

- Correction of the divergence: There exists $V_\ell \in C_c^\infty(B_2; \mathbb{R}^3)$ such that

$$\operatorname{div}(\tilde{U} + V_\ell) = 0\,,$$

$$V_\ell \to 0\,, \quad \text{in } C^k \text{ for all } k, \text{ as } \ell \to \infty\,.$$

- Final vortex ring: For any $\ell \gg 1$ we set

$$\bar{U}_\ell = \tilde{U} + V_\ell\,.$$

Step 3: Instability of the vortex ring

Theorem (Instability of $\bar{U}_\ell$)

There exists ℓ_0 such that, if $\ell \geq \ell_0$ then the linear operator

$$-\mathcal{L}_{\mathrm{ss},\ell} U = -\frac{1}{2}(1 + \xi \cdot \nabla) U - \Delta U + \mathbb{P}(\bar{U}_\ell \cdot \nabla U + U \cdot \nabla \bar{U}_\ell),$$

admits an unstable eigenvalue, i.e. there exist $\lambda_\ell \in \mathbb{C} \cap \{\mathrm{Re} > 0\}$ and $\eta_\ell \in H^k$ for any $k \geq 0$, such that $\mathcal{L}_{\mathrm{ss},\ell} \eta_\ell = \lambda_\ell \eta_\ell$.

- Since $\bar{U}_\ell$ is axisymmetric-no-swirl, we have

$$\mathcal{L}_{\mathrm{ss},\ell} : D(\mathcal{L}_{\mathrm{ss},\ell}) \subset L^2_{\mathrm{ans}} \to L^2_{\mathrm{ans}}$$

  and $\eta_\ell \in L^2_{\mathrm{ans}}$ as well.

Idea of proof

- First reduction: It is enough to prove instability for the operator

$$-\mathcal{L}_{\mathrm{st},\ell} U = \mathbb{P}(\bar{U}_\ell \cdot \nabla U + U \cdot \nabla \bar{U}_\ell)\,.$$

- Second reduction: We study the spectral problem in vorticity formulation

$$\mathcal{L}_{\mathrm{vor},\ell} : D(\mathcal{L}_{\mathrm{vor},\ell}) \subset L^2_{\mathrm{aps}} \to L^2_{\mathrm{aps}}\,.$$

- Key observation: When $\ell \gg 1$ the operator $\mathcal{L}_{\mathrm{vor},\ell}$ is “close” to

$$\mathcal{L}\omega = -\zeta(r)\partial_\theta \omega - (\mathrm{BS}_2[\omega] \cdot e_r) g'(r)\,.$$

First reduction

- Perturbative argument: If $\mathcal{L}_{\mathrm{st},\ell} U = -\mathbb{P}(\bar{U}_\ell \cdot \nabla U + U \cdot \nabla \bar{U}_\ell)$ admits an unstable eigenvalue, then there exists $\beta \ll 1$ such that

$$-\mathcal{L}_\beta U = -\beta \left(\frac{1}{2}(1 + \xi \cdot \nabla) U + \Delta U \right) + \mathbb{P}(\bar{U}_\ell \cdot \nabla U + U \cdot \nabla \bar{U}_\ell) \, .$$

- By replacing $\bar{U}_\ell$ with $\frac{1}{\beta} \bar{U}_\ell$, we get that $\mathcal{L}_{\mathrm{ss},\ell}$ admits an unstable eigenvalue.

Second reduction

- We set

$$\mathcal{L}_{\mathrm{vor},\ell} := \mathrm{curl} \circ \mathcal{L}_{\mathrm{st},\ell} \circ \mathrm{BS}_{3d} : D(\mathcal{L}_{\mathrm{vor},\ell}) \subset L^2_{\mathrm{aps}} \to L^2_{\mathrm{aps}} .$$

- We identify

$$L^2_{\mathrm{aps}}(\mathbb{R}^3) \sim L^2(\mathbb{R}_+ \times \mathbb{R}) , \quad -\omega(r,z)e_\theta \sim \omega(r,z) ,$$

hence,

$$\mathcal{L}_{\mathrm{vor},\ell} : D(\mathcal{L}_{\mathrm{vor},\ell}) \subset L^2(\mathbb{R}_+ \times \mathbb{R}) \to L^2(\mathbb{R}_+ \times \mathbb{R}) .$$

- $\mathrm{curl}\, \bar{U}_\ell = -\bar{\omega}_\ell e_\theta$, $\omega \in L^2(\mathbb{R}_+ \times \mathbb{R})$,

$$\begin{cases} -\mathcal{L}_{\mathrm{vor},\ell}\omega := (\bar{U}_\ell \cdot \nabla)\omega + (U \cdot \nabla)\bar{\omega}_\ell - \frac{\bar{U}^r_\ell}{r}\omega - \frac{U^r}{r}\bar{\omega}_\ell \\ U = \mathrm{BS}_{3d}[-\omega e_\theta] . \end{cases}$$

Convergence to Vishik's operator

$$\begin{cases} -\mathcal{L}_{\mathrm{vor},\ell}\omega := (\bar{U}_\ell \cdot \nabla)\omega + (U \cdot \nabla)\bar{\omega}_\ell - \frac{\bar{U}^r_\ell}{r}\omega - \frac{U^r}{r}\bar{\omega}_\ell \\ U = \mathrm{BS}_\ell[\omega]\,. \end{cases}$$

- $(\bar{U}_\ell \cdot \nabla)\omega \to (u \cdot \nabla)\omega$
- $\bar{\omega}_\ell \to \omega = \mathrm{curl}\, u$
- $-\frac{\bar{U}^r_\ell}{r}\omega - \frac{U^r}{r}\bar{\omega}_\ell \to 0$
- $\mathrm{BS}_\ell[\omega] \to \mathrm{BS}_{2d}[\omega]$

In conclusion:

$$\mathcal{L}_{\mathrm{vor},\ell}\omega \to \mathcal{L}\omega = -\zeta(r)\partial_\theta\omega - (\mathrm{BS}_2[\omega] \cdot e_r)g'(r)$$

Convergence of BS laws

- Let $\omega \in L^2(\mathbb{R}_+ \times \mathbb{R})$, we have

$$\mathrm{BS}_{3d}[-\omega e_\theta] = -\partial_z \psi \, e_r + \left(\partial_r + \frac{1}{r}\right) \psi \, e_z \,.$$

where, the stream function ψ is obtained by solving

$$\partial_r^2 \psi + \frac{1}{r}\partial_r \psi - \frac{1}{r^2}\psi + \partial_z^2 \psi = \omega \quad \text{in } \mathbb{R}_+ \times \mathbb{R} \,.$$

- Let $\omega \in L^2(\mathbb{R}^2)$, we have

$$\mathrm{BS}_{2d}[\omega] = -\partial_{x_2} \psi \, e_1 + \partial_{x_1} \psi \, e_2 \,.$$

where, the stream function ψ is obtained by solving

$$\partial_{x_1}^2 \psi + \partial_{x_2}^2 \psi = \omega \quad \text{in } \mathbb{R}^2 \,.$$

Gluing non-unique Navier-Stokes solutions

Gluing non-unique Navier-Stokes solutions

Theorem (Albritton-B.-Colombo ’22)

Let Ω be a smooth bounded domain, or $\mathbb{T}^3$. Then, there exist u and $\bar{u}$, two distinct suitable Leray-Hopf solutions to (NS) with identical body force $f \in L^1_t L^2_x$, initial conditions $u(\cdot, 0) = \bar{u}(\cdot, 0) = 0$, and no-slip boundary conditions.

- The non-uniqueness in $\mathbb{R}^3$ is driven by the instability of

$$\bar{u}(x, t) = \frac{1}{\sqrt{t}} \bar{U}\left(\frac{x}{\sqrt{t}}\right) .$$

- The non-uniqueness “emerges” from the irregularity at the space-time origin and is expected to be local.

Gluing non-unique Navier-Stokes solutions

Assume that $B_{1/2}(0) \subset \Omega$.

- For $t \ll 1$, we have

$$\operatorname{supp} \bar{u}(t, \cdot), \operatorname{supp} f(t, \cdot) \subset B_{1/2}(0) \subset \Omega\,.$$

  Hence, $(\bar{u}, f)$ satisfies (NS) in Ω with no-slip boundary conditions.

- The second solution $u = \bar{u} + u^{\mathrm{lin}} + u^{\mathrm{per}}$ is supported on the hole $\mathbb{R}^3$.
- Goal: We want to build a new solution v to (NS) in Ω, s.t.
  - v satisfies no-slip boundary conditions
  - $v \sim u$ in $B_{1/10}(0) \subset \Omega$

The main Ansatz

$$v(x,t) = \bar{u}(x,t) + \eta(x)\varphi(x,t) + \psi(x,t)\,, \quad x \in \Omega\,,\ t \in (0,t_0)\,.$$

- η is a smooth cut-off function, i.e.

$$\eta = 1 \quad \text{in } B_{1/9}(0)\,, \qquad \eta = 0 \quad \text{in } \Omega \setminus B_{1/7}(0)\,.$$

- φ is the inner solution, i.e. it will satisfy $\varphi \sim u$.
- ψ is the outer solution, it satisfies no-slip boundary conditions.

Inner and outer equations

We plug the Ansatz

$$v(x,t) = \bar{u}(x,t) + \eta(x)\varphi(x,t) + \psi(x,t)$$

into (NS) and we decouple the system as follows.

- Inner equation: In self-similar variables, $\xi = x/\sqrt{t}$, $\tau = \log(t)$,

$$\varphi(x,t) = \frac{1}{\sqrt{t}}\Phi(\xi,\tau)\,, \quad \psi(x,t) = \frac{1}{\sqrt{t}}\Psi(\xi,\tau)\,, \quad \eta(x) = N(\xi,\tau)$$

  We have

$$\begin{aligned} \partial_\tau \Phi - \mathcal{L}_{\mathrm{ss}}\Phi + \Phi\cdot\nabla(\Phi N) + \operatorname{div}(\tilde{N}\Psi\otimes\Phi + \tilde{N}\Phi\otimes\Psi) \\ + \bar{U}\cdot\nabla\Psi + \Psi\cdot\nabla\bar{U} + \nabla\Pi = 0\,, \end{aligned}$$

  where $\tilde{N}(\xi,\tau) = N(\xi/3,\tau)$.

- Outer equation:

$$\begin{aligned} \partial_t\psi - \Delta\psi + \psi\cdot\nabla\psi + \nabla\pi - \varphi\Delta\eta - 2\nabla\varphi\cdot\nabla\eta + (\psi\cdot\nabla\eta)\varphi = 0 \\ \operatorname{div}\psi = -\nabla\eta\cdot\varphi \end{aligned}$$

Inner equation

$$\partial_\tau \Phi - \mathcal{L}_{\mathrm{ss}}\Phi + \Phi \cdot \nabla(\Phi N) + \mathrm{div}(\tilde{N}\Psi \otimes \Phi + \tilde{N}\Phi \otimes \Psi) + \bar{U} \cdot \nabla\Psi + \Psi \cdot \nabla\bar{U} + \nabla\Pi = 0\,,$$

- We solve in the whole $\mathbb{R}^3$ using linear instability and the Ansatz

$$\Phi = \Phi^{\mathrm{lin}} + \Phi^{\mathrm{per}}\,.$$

- The orange terms are linear, but small.
- The purple terms are non-linear.

Outer equation

$$\partial_t\psi - \Delta\psi + \psi\cdot\nabla\psi + \nabla\pi - \varphi\Delta\eta - 2\nabla\varphi\cdot\nabla\eta + (\psi\cdot\nabla\eta)\varphi = 0$$
$$\operatorname{div}\psi = -\nabla\eta\cdot\varphi$$

- It amounts to solve the Navier-Stokes equation with inhomogeneous divergence and no-slip boundary conditions.
- The purple terms are treated as perturbations. Small because:
  - All the derivatives of η are concentrated on $A = B_{1/7}(0) \setminus B_{1/9}(0)$
  - Φ decays at infinity, hence $\varphi(x,t) = \frac{1}{\sqrt{t}}\Phi(x/\sqrt{t}, \log(t))$ is small in A, when $t \ll 1$.

Functional setting and fixed point argument

For $\bar{t} > 0$, $\bar{\tau} = \log(\bar{t})$, and $\alpha, \beta > 0$, we define the norms

$$\|\Phi\|_{X^{\alpha}_{\bar{\tau}}} := \sup_{\tau \le \bar{\tau}} e^{-\tau\alpha} \|\Phi(\cdot, \tau)\|_{L^{\infty}_{w}}$$

$$\|\psi\|_{Y^{\beta}_{\bar{t}}} := \sup_{s \in (0, \bar{t})} s^{-\beta} \|\psi(\cdot, s)\|_{L^{\infty}(\Omega)} .$$

- Fixed-point argument in the space

$$Z^{\alpha,\beta}_{\bar{t}} := X^{\alpha}_{\bar{\tau}} \times Y^{\beta}_{\bar{t}} .$$

- Key ingredients: Weighted semigroup estimate

$$\|e^{\tau \mathcal{L}_{\mathrm{ss}}} \mathbb{P} \operatorname{div} \|_{L^{p}_{w} \to L^{\infty}_{w}} \le C(p, \delta) \tau^{-(\frac{1}{2} + \frac{3}{2p})} e^{(a+\delta)\tau} ,$$

for any $p \in (3, +\infty]$, $\delta > 0$.

Conclusions

Weak solutions to the forced (NS) in the energy class, i.e. Leray solutions, are not unique.

- The non-uniqueness in $\mathbb{R}^3$ is driven by extreme the instability of a self-similar solution
$$\bar{u}(x,t) = \frac{1}{\sqrt{t}} \bar{U}\left(\frac{x}{\sqrt{t}}\right) ,$$
in full agreement with [Jia-Sverak' 14] and [Sverak-Guillod '18].

- The non-uniqueness "emerges" from the irregularity at the space-time origin shows a certain locality and robustness.

Open problems

- Is it possible to remove the force?
- There should be many unstable profiles $\bar{U}$. How generic are they? Is there an easier way to find them?
- Gluing non-unique solutions to the Euler equations?

Thank you for your attention!