Instability and non-uniqueness of Leray solutions of the forced Navier-Stokes equations, Part 2 [1]

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[1] Joint with D. Albritton and M. Colombo

Main result

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = f \\ \operatorname{div} u = 0 \\ u(\cdot, 0) = u_0 \end{cases} \qquad \text{on } \mathbb{R}^3 \times [0, T] \qquad \text{(NS)}$$

Theorem (Albritton-B.-Colombo '21, '22)

Let Ω be $\mathbb{R}^3$, a smooth bounded domain, or $\mathbb{T}^3$. Then, there exist u and $\bar{u}$, two distinct suitable Leray-Hopf solutions to (NS) with identical body force $f \in L^1_t L^2_x$ and $u(\cdot, 0) = \bar{u}(\cdot, 0) = 0$. When Ω is a bounded domain, u and $\bar{u}$ satisfy no-slip boundary conditions and f is supported far away from the boundary.

Strategy of proof when $\Omega = \mathbb{R}^3$

Self-similar structure

- There exists a div-free velocity field $\bar{U} \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ s.t.

$$\bar{u}(x,t) = \frac{1}{\sqrt{t}} \bar{U}\left(\frac{x}{\sqrt{t}}\right) .$$

$$\bar{u} \in C^\infty(\mathbb{R}^3 \times (0,T)) \cap C^0([0,T]; L^1 \cap L^{3-}) .$$

- There exists $F \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that

$$f(x,t) = \frac{1}{t^{3/2}} F\left(\frac{x}{\sqrt{t}}\right) .$$

$$f \in C^\infty(\mathbb{R}^3 \times (0,T)) \cap L^1([0,T]; L^1 \cap L^{3-}) .$$

The second solution

- We look for a second solution $u \neq \bar{u}$ to (NS) with body force f and $u(\cdot, 0) = 0$.
- To build the second solution, we need to choose a special background profile $\bar{U}$.
- Fundamental requirements:
  - $\bar{U}$ should decay sufficiently fast at ∞,
  - $\bar{U}$ is an unstable steady state for the (NS) in similarity variables.

Similarity variables

Let u be a solution to (NS) with body force f.

- Change of variables: $\xi = x/\sqrt{t}$, $\tau = \log(t) \in (-\infty, T)$

$$u(x,t) = \frac{1}{\sqrt{t}} U(\xi,\tau)$$

$$f(x,t) = \frac{1}{t^{3/2}} F(\xi)$$

- NS in similarity variables: $(\xi,\tau) \in \mathbb{R}^3 \times (-\infty, T)$

$$\partial_\tau U - \frac{1}{2}(1 + \xi \cdot \nabla) U - \Delta U + U \cdot \nabla U + \nabla P = F \qquad \text{(ssNS)}$$

Instability in (ssNS) generates non-uniqueness

- We think of $\bar{U}$ as a stationary solutions to (NS) in similarity variables with body force F

$$-\frac{1}{2}(1+\xi\cdot\nabla)\bar{U} - \Delta\bar{U} + \bar{U}\cdot\nabla\bar{U} + \nabla P = F\,.$$

- (Linear) Instability in similarity variables $\implies$ non-uniqueness.
- Heuristic: $\bar{U}$ is an unstable steady state if there exists $U(\xi,\tau)$ solving (ssNS) such that

$$\|U(\cdot,\tau) - \bar{U}(\cdot)\| \lesssim e^{a\tau}\,, \quad a > 0\,,\ \tau \in \mathbb{R}\,,$$

hence, setting $u(x,t) = \frac{1}{\sqrt{t}}U(\xi)$, we have

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\| = o(1)\,, \quad \text{as } t \to 0\,.$$

The linearized equation around $\bar{U}$

- $U = \bar{U} + V$ solves (ssNS) iff

$$\partial_\tau V = -\mathbb{P}(\bar{U} \cdot \nabla V + V \cdot \nabla \bar{U}) + \Delta V + \frac{1}{2}(1 + \xi \cdot \nabla)V - \mathbb{P}(V \cdot \nabla V)$$
$$= \mathcal{L}_{ss} V - \mathbb{P}(V \cdot \nabla V)\,, \qquad (\xi, \tau) \in \mathbb{R}^3 \times (-\infty, T)\,.$$

  where

$$-\mathcal{L}_{\mathrm{ss}} V = -\frac{1}{2}(1 + \xi \cdot \nabla)V - \Delta V + \mathbb{P}(\bar{U} \cdot \nabla V + V \cdot \nabla \bar{U})\,.$$

- Functional setting: $\mathcal{L}_{\mathrm{ss}} : D(\mathcal{L}_{\mathrm{ss}}) \subset L^2_\sigma \to L^2_\sigma$ is a closed operator, where

$$D(\mathcal{L}_{\mathrm{ss}}) := \{V \in L^2_\sigma \ : \ V \in H^2(\mathbb{R}^3)\,,\, \xi \cdot \nabla U \in L^2(\mathbb{R}^3)\}$$

Linear instability

Definition (Linear Instability)

We say that $\mathcal{L}_{\mathrm{ss}} : D(\mathcal{L}_{\mathrm{ss}}) \subset L^2_\sigma \to L^2_\sigma$ has an unstable eigenvalue if there exist

- $\lambda \in \mathbb{C}$ with $a := \mathrm{Re}\lambda > 0$
- $\eta \in H^k(\mathbb{R}^3; \mathbb{R}^3)$ for any $k > 0$, with $\mathrm{div}\ \eta = 0$

such that

$$\mathcal{L}_{\mathrm{ss}}\eta = \lambda\eta\,.$$

Linearized (ssNS)

Assume that $\mathcal{L}_{ss}$ has an unstable eigenvalue λ. Set

$$U^{\mathrm{lin}}(\xi,\tau) = \mathrm{Re}(e^{\lambda\tau}\eta(\xi))\,, \quad \xi \in \mathbb{R}^3\,, \tau \in \mathbb{R}\,.$$

- U^{lin} solves the linearized (ssNS)

$$\partial_\tau U^{\mathrm{lin}} = \mathcal{L}_{\mathrm{ss}} U^{\mathrm{lin}}\,, \quad \text{for any } \tau \in \mathbb{R}\,.$$

- Exponential growth:

$$|U^{\mathrm{lin}}(\cdot,\tau)| \sim e^{\mathrm{Re}\lambda\tau} = e^{a\tau}\,, \quad \tau \in \mathbb{R}\,.$$

From linear Instability to nonlinear instability

Assume that $\mathcal{L}_{ss}$ has an unstable eigenvalue λ. Set

$$U^{\mathrm{lin}}(\xi,\tau) = \mathrm{Re}(e^{\lambda\tau}\eta(\xi))\,, \quad \xi \in \mathbb{R}^3\,, \tau \in \mathbb{R}\,.$$

Theorem (Nonlinear instability)

Assume that λ is maximal unstable, i.e.

$$\sup_{z\in\sigma(\mathcal{L}_{\mathrm{ss}})} \mathrm{Re}\, z = \mathrm{Re}\lambda = a\,.$$

Then, there exist $T \in \mathbb{R}$ and a div-free vector field $U^{\mathrm{per}} : \mathbb{R}^3 \times (-\infty, T) \to \mathbb{R}^3$ such that

- *Regularity and decay:*

$$\|U^{\mathrm{per}}(\cdot,\tau)\|_{H^k} \lesssim e^{2a\tau}\,, \quad \tau \leq T,\, k \geq 0$$

- $V := U^{\mathrm{lin}} + U^{\mathrm{per}}$ *solves*

$$\partial_\tau V = \mathcal{L}_{\mathrm{ss}} V - \mathbb{P}(V\cdot\nabla V)\,.$$

From nonlinear instability to non-uniqueness

- $U := \bar{U} + U^{\rm lin} + U^{\rm per}$ solves (ssNS), i.e.

$$\partial_\tau U - \frac{1}{2}(1 + \xi \cdot \nabla)U - \Delta U + U \cdot \nabla U + \nabla P = F\,.$$

- Recall that

$$|\bar{U}(\cdot)| \sim 1\,, \quad |U^{\rm lin}(\cdot,\tau)| \sim e^{a\tau}\,, \quad |U^{\rm per}(\cdot,\tau)| \lesssim e^{2a\tau}\,, \quad \text{as } \tau \to -\infty\,,$$

hence,

$$|U(\cdot,\tau) - \bar{U}(\cdot)| = |U^{\rm lin}(\cdot,\tau) + U^{\rm per}(\cdot,\tau)| \sim e^{a\tau}\,, \quad \text{as } \tau \to -\infty\,.$$

From nonlinear instability to non-uniqueness

- We undo similarity variables (*ssNS*) $\to$ (*NS*)

$$\bar{u}(x,\cdot) = \frac{1}{\sqrt{t}}\bar{U}(\xi)\,,$$
$$u(x,t) = \frac{1}{\sqrt{t}}U(\xi,\tau)\,,$$

  where $\xi = x/\sqrt{t}$, $\tau = \log(t)$.

- We need to check that
  (a) $u \neq \bar{u}$
  (b) $u(\cdot,0) = \bar{u}(\cdot,0) = 0$

  Both (a) and (b) follow from $|U(\cdot,\tau) - \bar{U}(\cdot)| \sim e^{a\tau}$ as $\tau \to -\infty$:

$$|u(x,t) - \bar{u}(x,t)| = \frac{1}{\sqrt{t}}|U(\xi,\tau) - \bar{U}(\xi)| \sim \frac{1}{\sqrt{t}}e^{a\tau} = t^{a-1/2} \to 0\,.$$

Resume

Theorem (Albritton-B.-Colombo '21)

There exist two distinct suitable Leray-Hopf solutions to (NS) with identical body force $f \in L^1_t L^2_x$ and $u(\cdot, 0) = \bar{u}(\cdot, 0) = 0$.

- Linear instability: There exists $\bar{U} \in C_c^\infty$ such that

$$\mathcal{L}_{\rm ss} : D(\mathcal{L}_{\rm ss}) \subset L^2_\sigma \to L^2_\sigma$$

  has a maximal unstable eigenvalue.

- Nonlinear instability: The unstable eigenvalue can be perturbed to $\bar{U}$, an unstable trajectory for (ssNS). In standard variables $u(x,t) = \frac{1}{\sqrt{t}} U(\xi)$ provides a second solution to (NS) with body force f and $u(0, \cdot) = 0$.

Theorem (Linear instability)

There exists a divergence-free vector field $\bar{U} \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ s.t. $\mathcal{L}_{ss} : D(\mathcal{L}_{ss}) \subset L^2_\sigma(\mathbb{R}^3; \mathbb{R}^3) \to L^2_\sigma(\mathbb{R}^3; \mathbb{R}^3)$

$$-\mathcal{L}_{\mathrm{ss}} V = -\frac{1}{2}(1 + \xi \cdot \nabla) V - \Delta V + \mathbb{P}(\bar{U} \cdot \nabla V + V \cdot \nabla \bar{U})$$

has a maximal unstable eigenvalue.

Theorem (Nonlinear instability)

Set $U^{\mathrm{lin}}(\xi, \tau) = \mathrm{Re}(e^{\lambda \tau} \eta(\xi))$. There exist $T \in \mathbb{R}$ and a div-free vector field $U^{\mathrm{per}} : \mathbb{R}^3 \times (-\infty, T) \to \mathbb{R}^3$ such that

- *Regularity and decay:*

$$\|U^{\mathrm{per}}(\cdot, \tau)\|_{H^k} \lesssim e^{2a\tau}, \quad \tau \leq T,\, k \geq 0$$

- *$V := U^{\mathrm{lin}} + U^{\mathrm{per}}$ solves*

$$\partial_\tau V = \mathcal{L}_{\mathrm{ss}} V - \mathbb{P}(V \cdot \nabla V)\,.$$

Nonlinear instability: Heuristic

- We think of (ssNS) as an ODE in the Hilbert space $H = L^2_\sigma$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\tau} U(\tau) = \mathbf{b}(U(\tau)),$$

  where $\mathbf{b} : H \to H$ is the velocity field.

- $\bar{U} \in H$ is an equilibrium point, i.e.

$$\mathbf{b}(\bar{U}) = 0.$$

- Under our assumptions, the linearized operator

$$D\mathbf{b}(\bar{U}) = \mathcal{L}_{\mathrm{ss}},$$

  has an unstable eigenvalue.

We decompose

$$H = E^u \times E^0 \times E^s \,,$$

where E^u, E^0 and E^s are $D\mathbf{b}(\bar{U})$-invariant, and

- E^u is the collection of unstable directions, i.e $\sigma(D\mathbf{b}(\bar{U})|_{E^u}) \subset \{\mathrm{Re}\lambda > 0\}$
- E^0 is the central manifold, i.e. $\sigma(D\mathbf{b}(\bar{U})|_{E^0}) \subset \{\mathrm{Re}\lambda = 0\}$
- E^s is the collection of stable directions, i.e $\sigma(D\mathbf{b}(\bar{U})|_{E^s}) \subset \{\mathrm{Re}\lambda < 0\}$

Under our assumption $E^u \neq \{0\}$.

Theorem (Unstable Manifold)

Assume $H = \mathbb{R}^d$ and $E^u \neq \{0\}$. There exists a submanifold $M^u \subset H$ s.t.

- $\mathrm{Tan}_{\bar{U}} M^u = E^u$
- *For any $U_0 \in M^u$, it holds*

$$\lim_{\tau \to -\infty} U(\tau) = \bar{U}$$

where U solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau} U(\tau) = \mathbf{b}(U(\tau)) \\ U(0) = U_0 \end{cases}$$

- Stronger: It builds the entire unstable manifold. We need only one trajectory.
- Weaker: We need a much quantitative conclusion
  - Exponential decay at $\tau = -\infty$
  - Approximation with the solution to the linearized problem

    $$\frac{\mathrm{d}}{\mathrm{d}\tau} U(\tau) = D\mathbf{b}(\bar{U})[U(\tau)]\,.$$

    Here is where we used maximality.
- Technical point: We need a version of the unstable manifold theorem for infinite dimensional Hilbert spaces and unbounded vector fields. See for instance [Henry '81].

Nonlinear instability: Idea of proof

- Using that $U^{\rm lin} + U^{\rm per}$ solves the linearized (ssNS) we get

$$\partial_\tau U^{\rm per} = \mathcal{L}_{\rm ss} U^{\rm per} - \mathbb{P}(U^{\rm lin}\cdot\nabla U^{\rm per} + U^{\rm per}\cdot\nabla U^{\rm lin}) - \mathbb{P}(U^{\rm per}\cdot\nabla U^{\rm per}) - \mathbb{P}(U^{\rm lin}\cdot\nabla U^{\rm lin})$$

- Duhamel's formula:

$$U^{\rm per} = L(U^{\rm per}) + B(U^{\rm per}, U^{\rm per}) + G$$

where

$$L(U)(\cdot,\tau) = -\int_{-\infty}^{\tau} e^{(\tau-s)\mathcal{L}_{\rm ss}}\mathbb{P}(U\cdot\nabla U^{\rm per} + U^{\rm per}\cdot\nabla U)(\cdot,s)\,ds$$

$$B(U,U)(\cdot,\tau) = -\int_{-\infty}^{\tau} e^{(\tau-s)\mathcal{L}_{\rm ss}}\mathbb{P}(U\cdot\nabla U)(\cdot,s)\,ds$$

$$G(\cdot,\tau) = -\int_{-\infty}^{\tau} e^{(\tau-s)\mathcal{L}_{\rm ss}}\mathbb{P}(U^{\rm lin}\cdot\nabla U^{\rm lin})(\cdot,s)\,ds$$

Nonlinear instability: Idea of proof

We need to find a fixed point for the operator

$$\mathcal{T}(U) = L(U) + B(U,U) + G$$

Proposition (fixed point)

Let $N > 5/2$, $a = \mathrm{Re}\lambda$, $\varepsilon_0 \ll 1$ and $T < 0$. Set

$$\|U\|_X = \sup_{\tau \le T} e^{-\tau(a+\varepsilon_0)} \|U(\cdot,\tau)\|_{H^N(\mathbb{R}^3)} .$$

For T small enough $\mathcal{T} : X \to X$ is a contraction.

Key ingredients:

- Growth estimate: $\forall\, \delta > 0$ it holds

$$\|e^{\tau \mathcal{L}_{\mathrm{ss}}}\|_{H^N \to H^N} \le C(\delta, N) e^{(a+\delta)\tau} .$$

- Parabolic regularization: $\forall\, \delta > 0$ it holds

$$\|e^{\tau \mathcal{L}_{\mathrm{ss}}}\|_{L^2 \to H^1} \le C(\delta, N) \frac{1}{\tau^{1/2}} e^{(a+\delta)\tau} .$$

Seeking for Linear Instability

Theorem (Albritton-B.-Colombo)

There exists a divergence-free vector field $\bar{U} \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ s.t. the linear operator $\mathcal{L}_{ss} : D(\mathcal{L}_{ss}) \subset L^2_\sigma(\mathbb{R}^3;\mathbb{R}^3) \to L^2_\sigma(\mathbb{R}^3;\mathbb{R}^3)$

$$-\mathcal{L}_{\rm ss} U = -\frac{1}{2}(1+\xi\cdot\nabla)U - \Delta U + \mathbb{P}(\bar{U}\cdot\nabla U + U\cdot\nabla\bar{U})$$

has an unstable eigenvalue.

Definition (Linear Instability)

We say that $\mathcal{L}_{\rm ss} : D(\mathcal{L}_{\rm ss}) \subset L^2_\sigma \to L^2_\sigma$ has an unstable eigenvalue if there exist

- $\lambda \in \mathbb{C}$ with $a := {\rm Re}\lambda > 0$
- $\eta \in H^k(\mathbb{R}^3;\mathbb{R}^3)$ for any $k > 0$, with div $\eta = 0$

such that

$$\mathcal{L}_{\rm ss}\eta = \lambda\eta\,.$$

Strategy of proof

We appeal to the unstable vortex build in [Vishik '18]. The latter is an unstable steady state of the $2d$-Euler equations with power-law decay at ∞.

We aim to lift the unsteble vortex to a $3d$-unstable vortex ring with bounded support.

- Reduction to the Euler equations in standard variables
- Axisymmetric-no-swirl structure and vortex ring construction

Comparison with Jia-Sverak

- Our vortex ring $\bar{U}$ does not solve exactly (ssNS). It produces a body force F.
- Our unstable profile decays fast at infinity allowing for technical simplifications:
  - No need to cut-off the non-unique solutions
  - We only need to prove instability, instead of bifurcation scenarios.

2D Instability

2d-vorticity formulation: $\omega(x) = \mathrm{curl}\, u(x)$, $x \in \mathbb{R}^2$

$$\partial_t \omega + u \cdot \nabla \omega = \mathrm{curl}\, f\,,$$

- Shear flows: $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\bar{u}(x) = (b(x_2), 0)\,,$$

- Vortices: $x \in \mathbb{R}^2$, $r = |x|$,

$$\bar{u}(x) = \zeta(r) x^{\perp}\,, \quad \bar{\omega}(x) = g(r)\,.$$

2D Instability

- Linearized Euler equations around a vortex

$$\partial_t \omega + \zeta(r)\partial_\theta \omega + (\mathrm{BS}_2[\omega] \cdot e_r) g'(r) = 0 \,.$$

- Spectral problem

$$\mathcal{L}_{\mathrm{st}}\omega = -\zeta(r)\partial_\theta \omega - (\mathrm{BS}_2[\omega] \cdot e_r) g'(r) \,.$$

  - Instability: $\mathcal{L}_{\mathrm{st}}\omega = \lambda\omega$ where $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda > 0$.
  - Rayleigh's stability criterion: If $g'(r) < 0$ for all $r > 0$, then there are no unstable eigenvalues [Rayleigh '1880].

- Dimensional reduction: the following spaces are $\mathcal{L}_{\mathrm{st}}$-invariant

$$U_k(\mathbb{R}^2) = \{\omega \in L^2(\mathbb{R}^2) \;:\; \omega = f(r)e^{ik\theta}\} \,.$$

2d Instability

The eigenvalue problem $\mathcal{L}_{\mathrm{st}}\omega = \lambda\omega$ reduces to the Rayleigh's stability equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} - k^2\right)\varphi(s) - \frac{A(s)}{\Xi(s) - c}\varphi(s) = 0\,, \quad s \in \mathbb{R}\,.$$

- $s = \log(r)$, exponential coordinates
- $\varphi(s)e^{ik\theta}$ stream function
- A and Ξ are functions of g' and ζ, respectively
- $\lambda = -ick$

[Tollmien '34], [Lin '02], [Fadeev '71].

2D Instability: Vishik's theorem

Set

$$L^2_m(\mathbb{R}^2) := \bigotimes_{k=1}^{\infty} U_{km} = \{m\text{-fold symmetric functions}\}\,.$$

Theorem (Vishik '18, ABCDGJK'21)

There exists a smooth decaying vortex

$$\bar{u}(x) = \zeta(r)x^{\perp}\,, \qquad \bar{\omega}(x) = g(r)\,,$$

such that $\mathcal{L}_{st} : D(\mathcal{L}_{st}) \subset L^2_m(\mathbb{R}^2) \to L^2_m(\mathbb{R}^2)$*,* $m \geq 2$*, has an unstable eigenvalue.*

Sharpness of Yudovich class

Theorem (Vishik'18, ABCDGJK'21)

For every $p \in (2, \infty)$, there exist two distinct finite-energy weak solutions u and $\bar{u}$ of the $2d$-Euler equations with identical body force f such that

- $\omega, \bar{\omega} \in L^\infty_t (L^p_x \cap L^1_x)$;
- $f \in L^1_t L^2_x$ *and* $\mathrm{curl} f \in L^1_t (L^p_x \cap L^1_x)$.

how to build an unstable 3D-vortex ring

- Step 1: We truncate $\bar{u}$ to get an unstable, compactly supported vortex
- Step 2:We use the truncate vortex as a radial profile of $3D$-Axisymmetric-no-swirl velocity field
- Step 3: We employ spectral perturbative argument to show that the vortex ring inherits the instability of Vishik's vortex

What's next

- I'll present details of the construction of the unstable vortex ring
- I'll explain how to build non-uniqueness when Ω is a bounded domain, or a torus (gluing technique)
- I'll present open questions and related problems

Thank you for your attention!